Viscosity and Elliptic Flow

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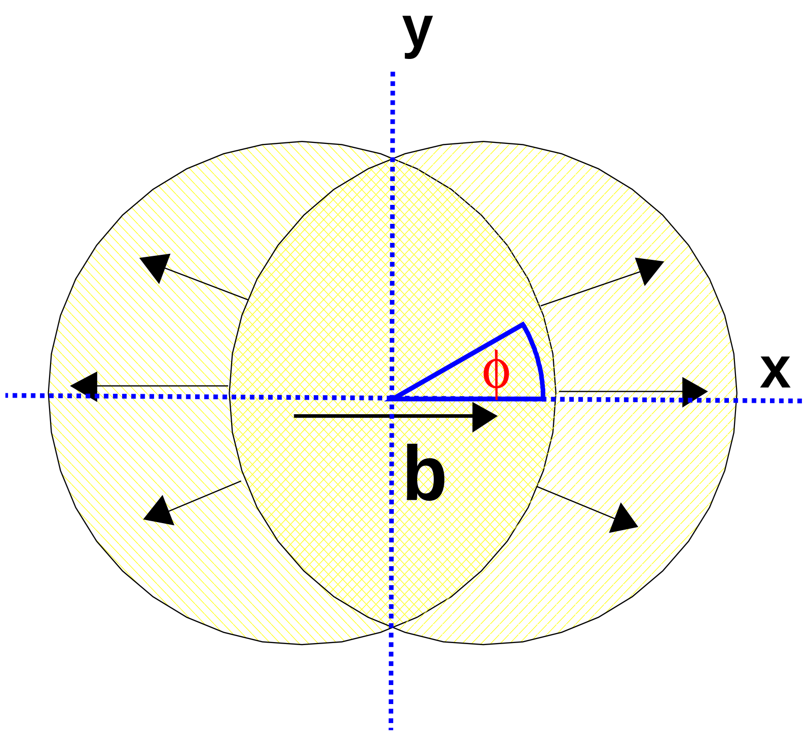
- Work done in collaboration with Kevin Dusling hep-ph/0710.5932

Outline

1. Viscosity of Heavy Ion Collisions
2. Remarks about relativistic viscous hydro
3. Solve viscous hydro in 2+1 dimensions with Bjorken symmetry.
4. Show the important effects for Heavy Ion Collisions
5. Discuss Limitations

Observation:



There is a large momentum anisotropy:

$$v_2 \equiv \frac{\langle p_x \rangle^2 - \langle p_y \rangle^2}{\langle p_x \rangle^2 + \langle p_y \rangle^2} \approx 20\%$$

Interpretation

- The medium responds as a fluid to differences in X and Y pressure gradients
- Hydrodynamic models work well enough.

Is the system *Large* enough? Does it live *Long* enough for hydro?

How *Long* and *Large* is Long/Large Enough ?

- Need the mean free path times expansion rate less than one

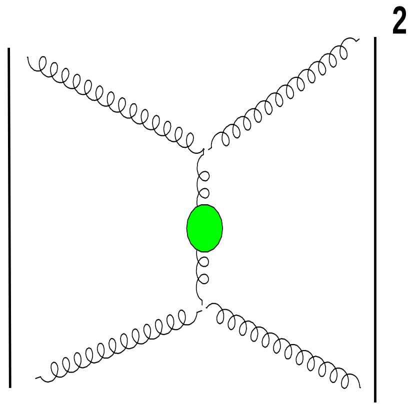
$$\ell_{\text{m.f.p.}} \times \text{Expansion Rate} \ll 1$$

How *Long* and *Large* is Enough ?

- Quick estimate of the mean free path:

$$\ell_{\text{m.f.p}} \equiv \frac{1}{n\sigma} = \frac{1}{\underbrace{n}_{\sim T^3} \times \underbrace{\sigma}_{\alpha_s^2/T^2}}$$

$$\sim \frac{1}{\alpha_s^2 T}$$



- So the Figure of Merit:

$$\overbrace{\ell_{\text{m.f.p.}}}^{1/\alpha_s^2 T} \times \overbrace{\text{expansion rate}}^{1/\tau} \ll 1$$

$$\underbrace{\frac{1}{\alpha_s^2}}_{\text{Liquid Parameter}} \times \underbrace{\frac{1}{\tau T}}_{\text{Experimental Parameter}} \ll 1$$

How *Long* and *Large* is *Long/Large* Enough ?

- What is the mean free path? $\ell_{mfp} \equiv \frac{\eta}{e+p}$
- The mean free path should be less than the expansion rate $\frac{1}{\tau}$:

$$\underbrace{\frac{\eta}{e+p}}_{\ell_{mfp}} \frac{1}{\tau} \quad \ll \quad 1$$

- Then using the relation: $(e+p) = sT$.

$$\underbrace{\frac{\eta}{s}}_{\text{Liquid parameter}} \times \underbrace{\frac{1}{\tau T}}_{\text{Experimental parameter: } \sim 1} \ll 1$$

1. η/s needs to be small to have interacting QGP at RHIC.
2. Even if η/s is small, dissipative effects are significant!

Estimates of η/s for the initial stage of the QGP

1. Perturbative QCD – Kinetic Theory — Arnold, Moore, Yaffe.

   $\eta \approx 150\, T^3 \frac{1}{g^4}$. Based upon kinetic theory of quarks and gluons. Set $\alpha_s \to 1/2$ and $m_D \to$ a reasonable value

$$\left(\frac{\ell_{mfp}}{\tau}\right) \approx \underbrace{0.3}_{\eta/s} \; \underbrace{\frac{1}{\tau T}}_{\sim 1}$$

$\ell_{mfp} \approx 4$ thermal wavelengths

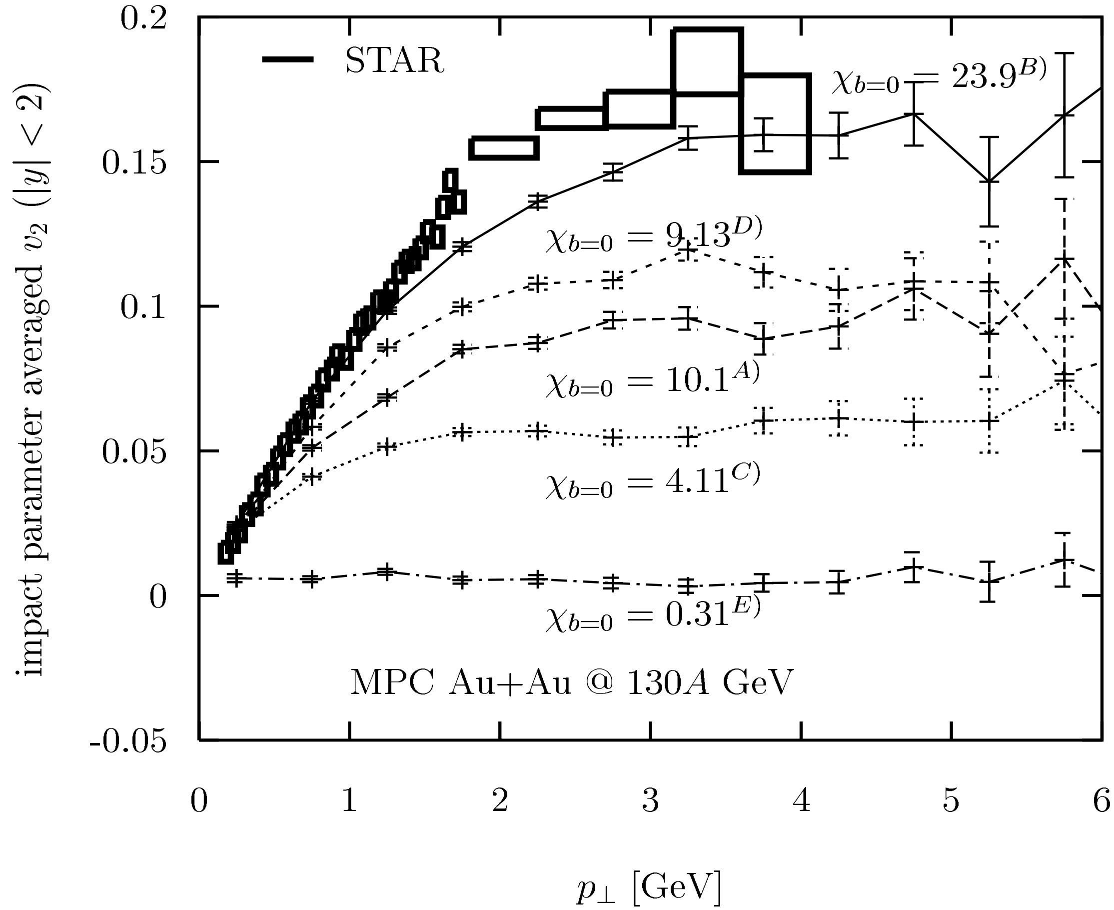
2. Strongly Coupled conformal N=4 SYM – AdS/CFT — Son, Starinets, Policastro

   No kinetic theory exists.

$$\left(\frac{\ell_{mfp}}{\tau}\right) = \underbrace{\frac{1}{4\pi}}_{\eta/s} \; \underbrace{\frac{1}{\tau T}}_{\sim 1}$$

$\ell_{mfp} \approx 1$ thermal wavelength

With these sorts of numbers (not weakly coupled) expect some collectivity.



- Classical Massless Particles with Constant Cross Section

$$\frac{\eta}{s} \sim \frac{1}{4\pi}$$

Summary at time τ_0

$$T_o \sim 300\,\text{MeV} \quad \text{and} \quad \tau_0 \sim 1\,\text{fm}$$

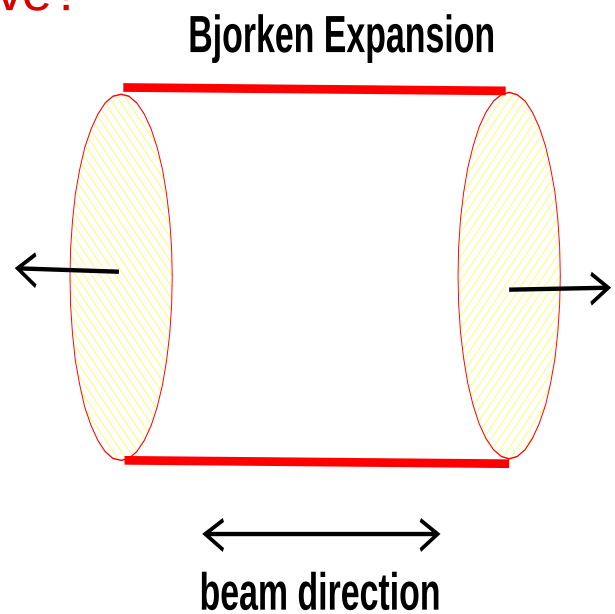
- Find:

$$\left(\frac{\Gamma_s}{\tau}\right) \approx 0.1 - 0.4$$

How does $\frac{\Gamma_s}{\tau}$ evolve?

- 1D Expansion – scales set by temperature.
- 3D Expansion – scales fixed.

How does Γ_s/τ evolve?



- 1D Bjorken Expansion – scales set by temperature
  - Temperature decreases $T \sim \frac{1}{\tau^{1/3}}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\#}{\tau T} \sim \# \frac{1}{\tau^{2/3}}$$

Viscous effects get steadily smaller

Viscous corrections to Ideal Hydrodynamics and Longitudinal Expansion

$$T^{ij} = p\delta^{ij} + \eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\,\partial_l v^l\right)$$

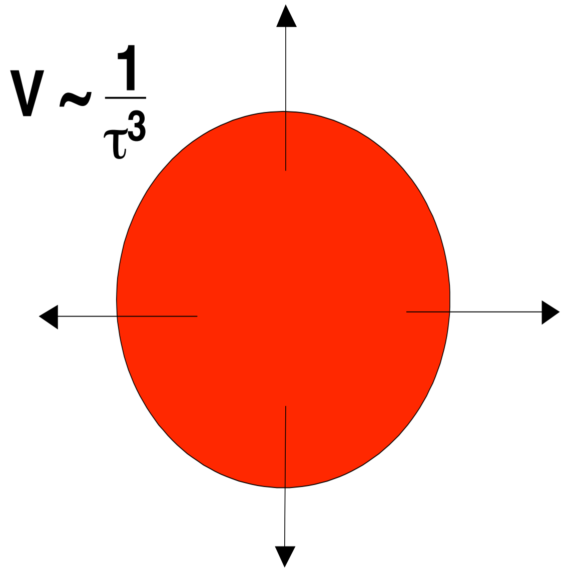
For a Bjorken expansion we have: $T^{zz}_{vis} \sim \eta\partial^z v^z \sim -\frac{\eta}{\tau}$

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_o + T^{\mu\nu}_{vis} \\ &= \begin{pmatrix} \epsilon & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & \frac{2}{3}\frac{\eta}{\tau} & & \\ & & \frac{2}{3}\frac{\eta}{\tau} & \\ & & & -\frac{4}{3}\frac{\eta}{\tau} \end{pmatrix} \end{aligned}$$

- The Longitudinal Pressure is reduced by $\frac{4}{3}\eta/\tau$.
- The Transverse Pressure is increased by $\frac{2}{3}\eta/\tau$.

Expect p_T spectra to be pushed out to larger p_T In a Radially Symmetric way

How does Γ_s/τ evolve?



- 3D Expansion – scales fixed
  - Density decreases $n \sim \frac{1}{\tau^3}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\#}{\tau n \sigma_o} \sim \# \frac{\tau^2}{\sigma_o}$$

Viscous effects get rapidly larger

Solving the Relativistic Navier Stokes Equations RNSE

- The RNSE as written can not be solved. There are unstable modes which propagate faster than the speed of light.

- Why? Because the stress RNSE tensor is not allowed time to change.

$$T^{ij}_{vis}\Big|_{\text{instantly}} = \eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_i v^i\right)$$

- Can make many models which relax to the RNSE.

$$T^{ij}_{vis}\Big|_{\omega\to 0} \sim \eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_i v^i\right)$$

- In the regime of validity of hydrodynamics the models all agree with each other and with RNSE.

Can solve these models

Relaxation Time Approximation

- Bjorken Expansion – Normal Viscous Hydro

$$\frac{de}{d\tau} = -\frac{e + T^{zz}}{\tau} \qquad T^{zz}_{eq} = p - \overbrace{\frac{4}{3}\frac{\eta}{\tau}}^{\partial_z u^z}$$

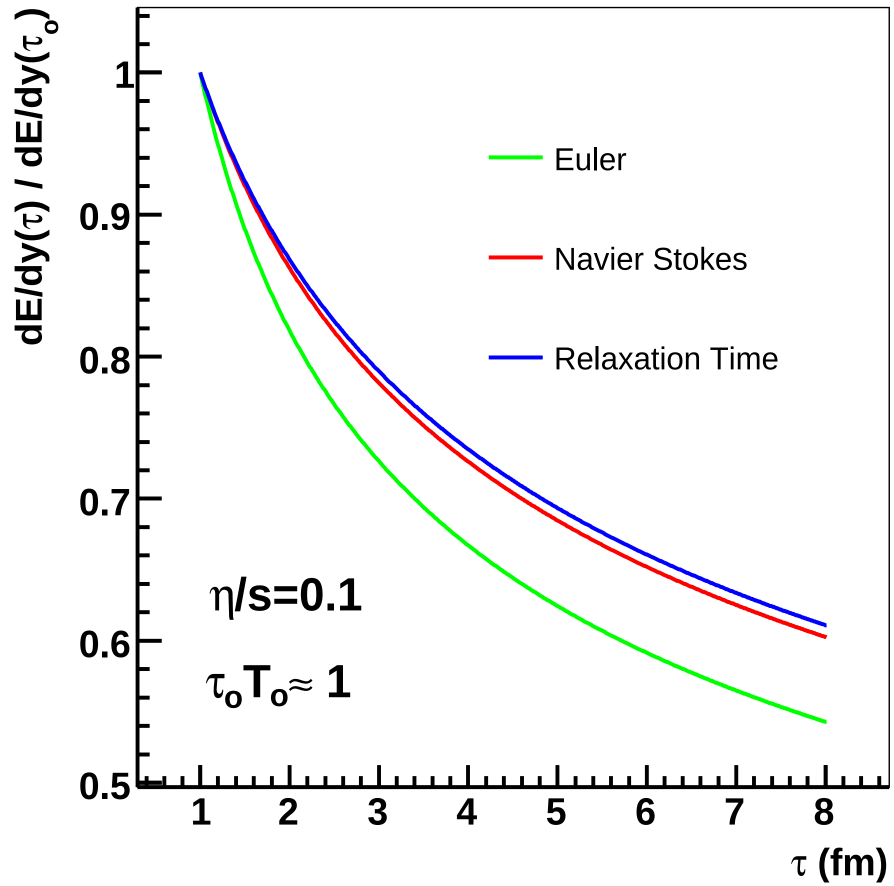
- Bjorken Expansion – Relaxation Time Approximation

$$\frac{de}{d\tau} = -\frac{e + T^{zz}}{\tau} \qquad \text{and} \qquad \frac{dT^{zz}}{d\tau} = -\frac{(T^{zz} - T^{zz}_{eq})}{\tau_R}$$

  - What are the appropriate initial conditions for this second equation?

Answer: $T^{zz} \simeq T^{zz}_{eq}$

Solution of Relaxation Time Equations



Relaxation is practically the same as Navier Stokes

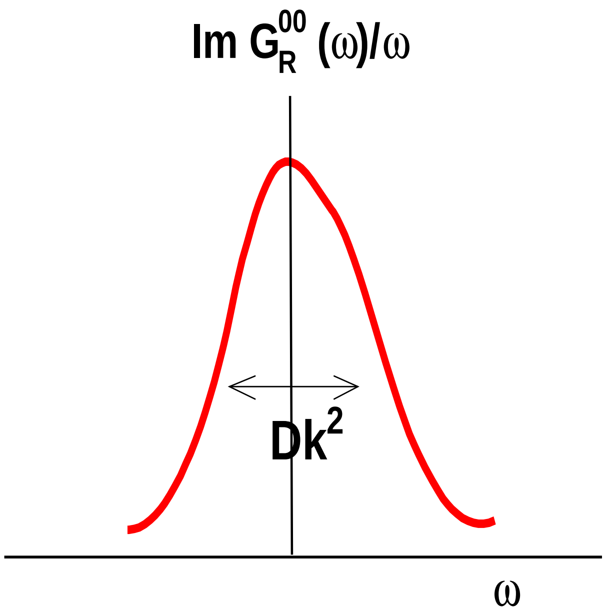
Made precise – L. Lindblom

Diffusion Equation

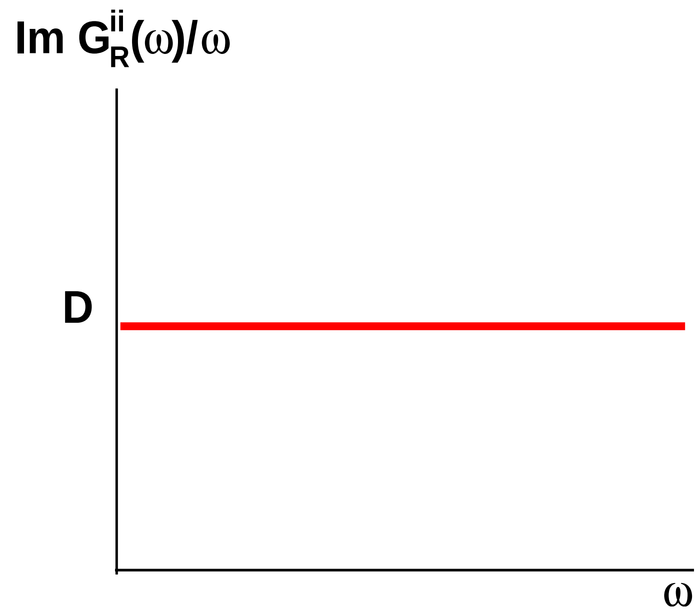
$$\partial_t n - D\nabla^2 n = 0$$

- Specifies the form of the spectral density at small k and ω

$$G_R(\omega, k) = \frac{1}{\partial_t - D\nabla^2} = \frac{1}{-i\omega + Dk^2}$$



$\Longrightarrow$

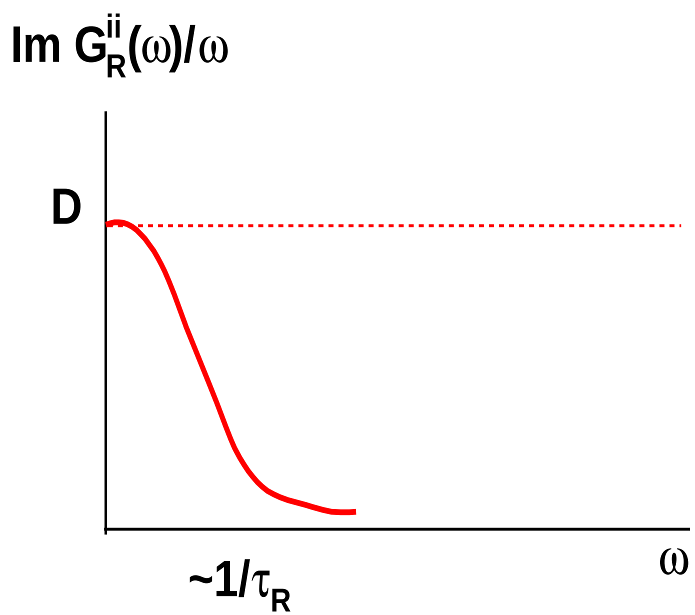


Relaxation Time Approximation:

$$\begin{aligned}\partial_t n + \partial_x j &= 0\\ \partial_t j &= -\frac{(j + D\nabla n)}{\tau_R}\end{aligned}$$

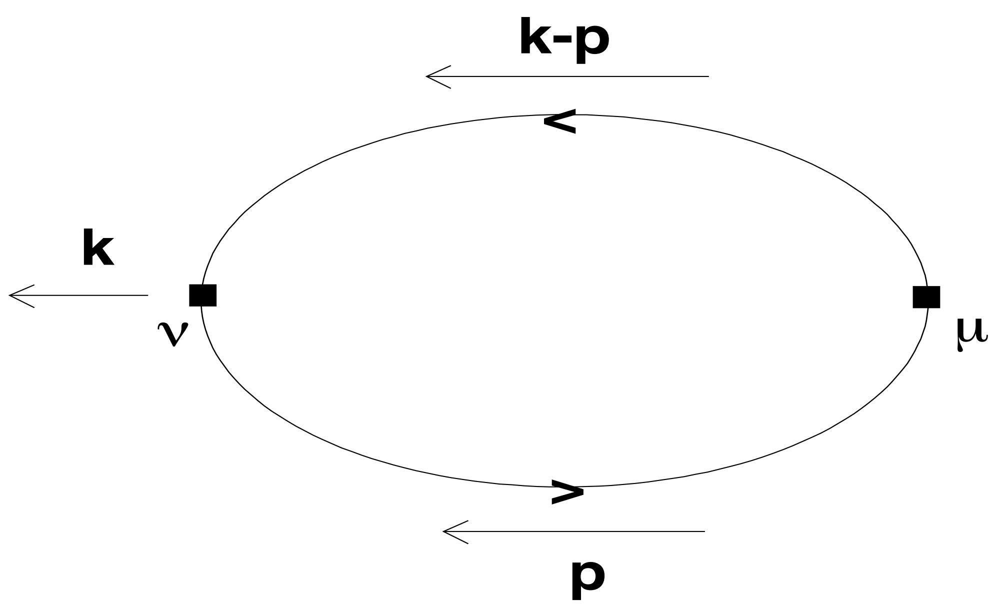
- Solve the system equations and find the retarded correlator

$$\frac{\mathrm{Im} G_R(\omega)}{\omega} = \frac{D}{\pi}\frac{1}{1+(\omega\tau_R)^2}$$



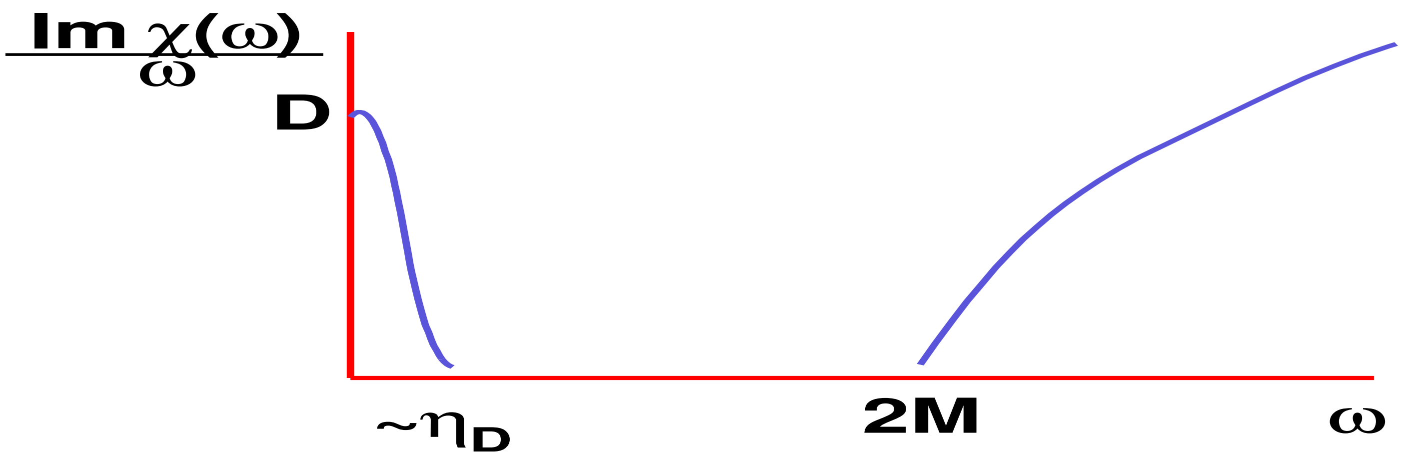
Spectral weight for a free theory:

$$\int e^{+i\omega t - i\mathbf{k}\cdot\mathbf{x}} \left\langle [J^i(\mathbf{x},t) J^i(0,0)] \right\rangle$$



Free Spectral Function:

$$\rho(\omega) = \underbrace{\frac{N_c}{8\pi^2}\,\omega^2\sqrt{1-\frac{4M^2}{\omega^2}}\left(2+\frac{4M^2}{\omega^2}\right)}_{\text{Vacuum}} + \underbrace{\chi_s\frac{T}{M}\,\omega\,\delta(\omega)}_{\text{Thermal}}$$



- Interactions will smear the delta function:

$$\delta(\omega) \to \frac{\eta_D}{\omega^2+\eta_D^2} \qquad\qquad \eta_D = \frac{T}{MD}$$

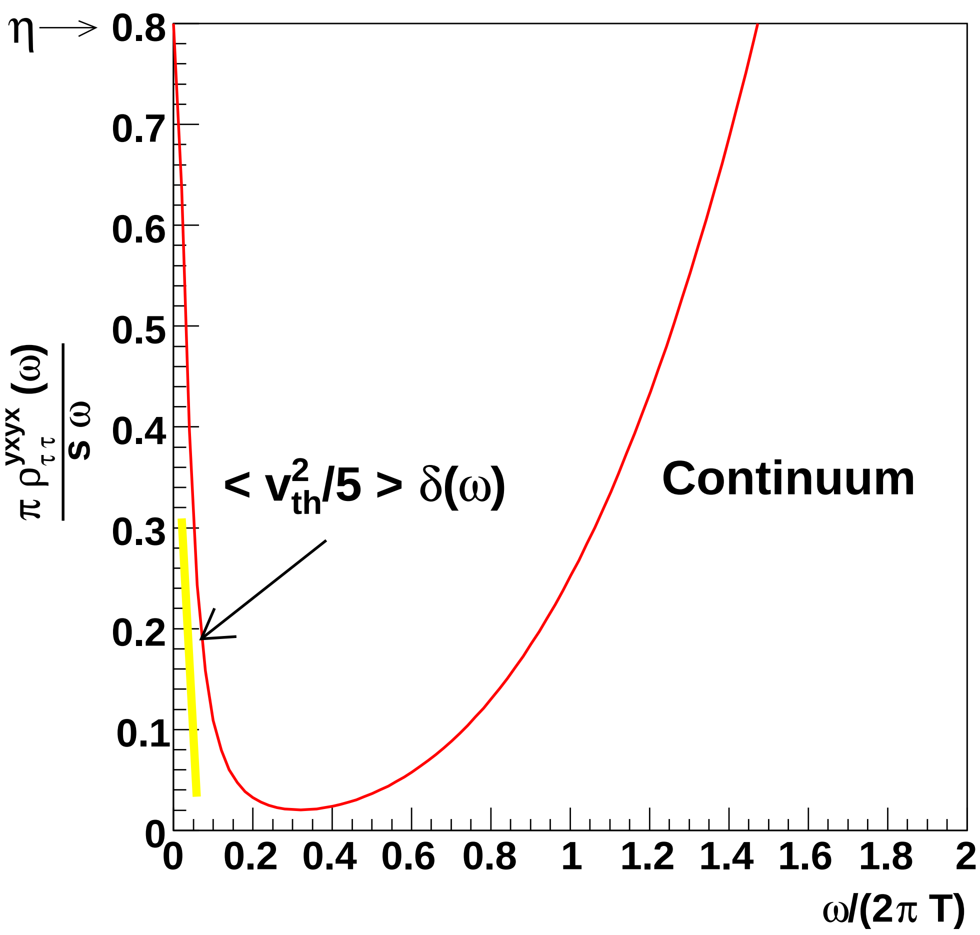
- The total integral under the delta function is constant:

$$\chi_s \underbrace{\frac{T}{M}}_{(\text{Thermal velocity})^2} \Longrightarrow \text{Independent of Interaction}$$

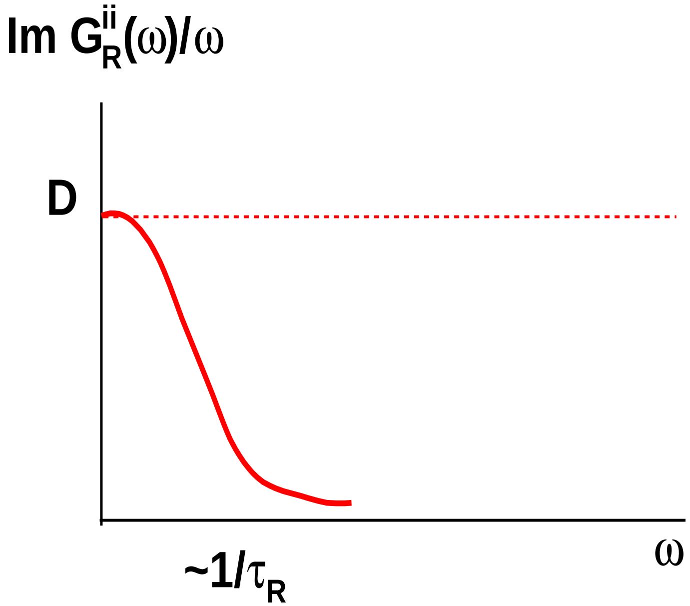
Real Spectral Densities:

- Relaxation models are a one parameter ansatz for the spectral density at small frequency which satisfy the f-Sum Rule

Cartoon of Weak Coupling



Weak Coupling Sum Rules and Short Time Response



- f-Sum Rule at Weak Coupling

$$\underbrace{\int d\omega\, \frac{\mathrm{Im} G_R^{ii}(\omega)}{\omega}}_{\text{Short Times}} = \left\langle v_{\rm th}^2 \right\rangle$$

- Substitute $\frac{G_R(\omega)}{\omega} \propto \frac{1}{1+(\omega\tau_R)^2}$

$$\underbrace{\frac{D}{\tau_R}}_{\text{Short Times}} = \left\langle v_{\rm th}^2 \right\rangle$$

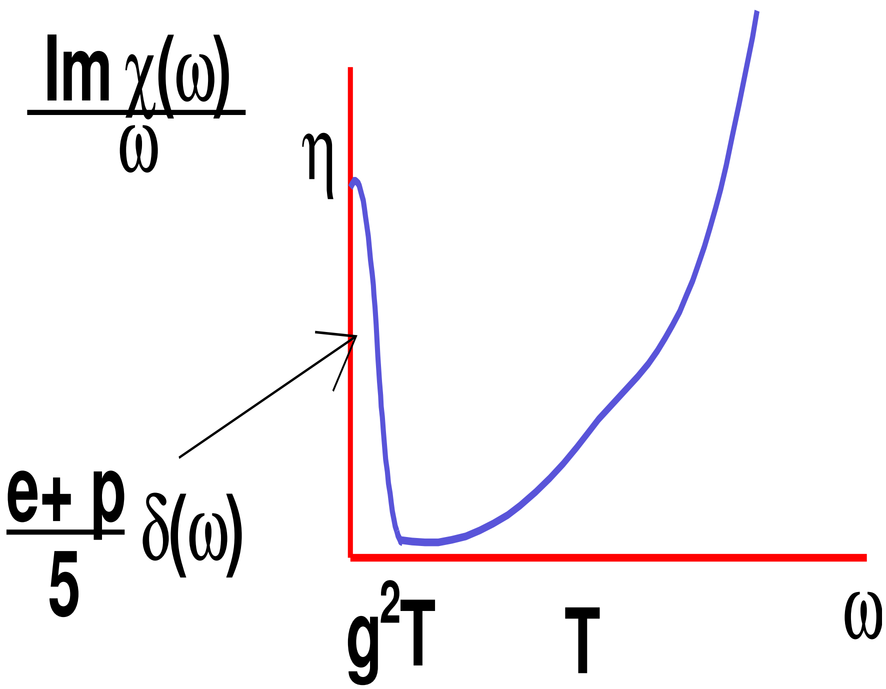
Use short and long time parameters:

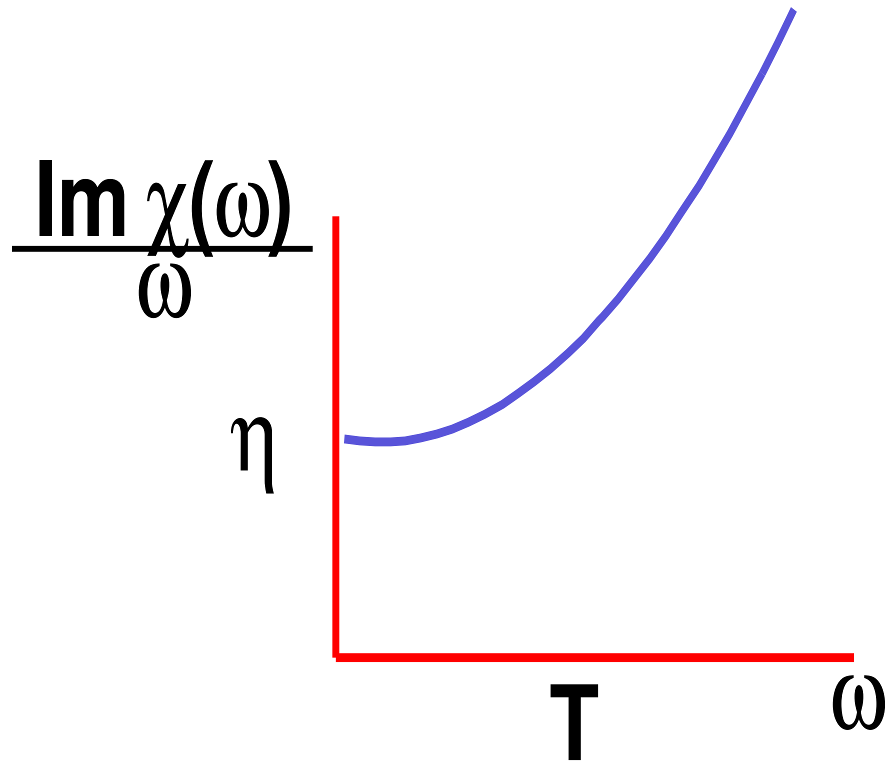
$$\begin{aligned} \partial_t n + \partial_x j &= 0 \\ \partial_t j &= -\frac{(j + D\nabla n)}{\tau_R} \end{aligned}$$

- Long Time Parameters: D
- Short Time Parameters: $\frac{D}{\tau_R} = \langle v_{\mathrm{th}}^2 \rangle$
- Results should (and will!) be insensitive to short time response

Shear Visocisty and Strong Coupling:

$$\chi(k,\omega) = \int e^{+i\omega t - \mathbf{k}\cdot\mathbf{x}} \langle [T^{xy}(t), T^{xy}(0)] \rangle$$



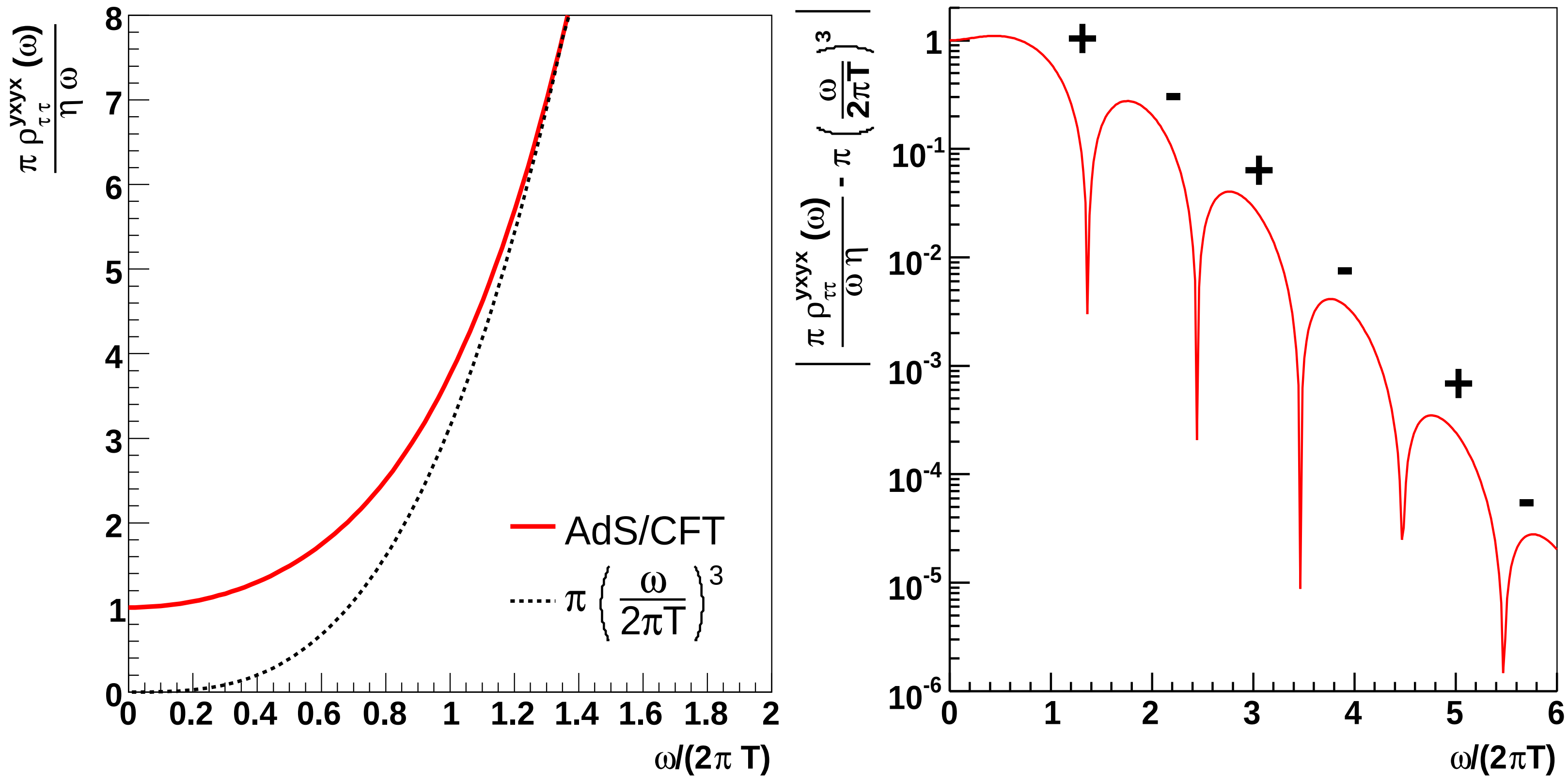


What happens at strong coupling?

Strong Coupling and the AdS/CFT Correspondence:

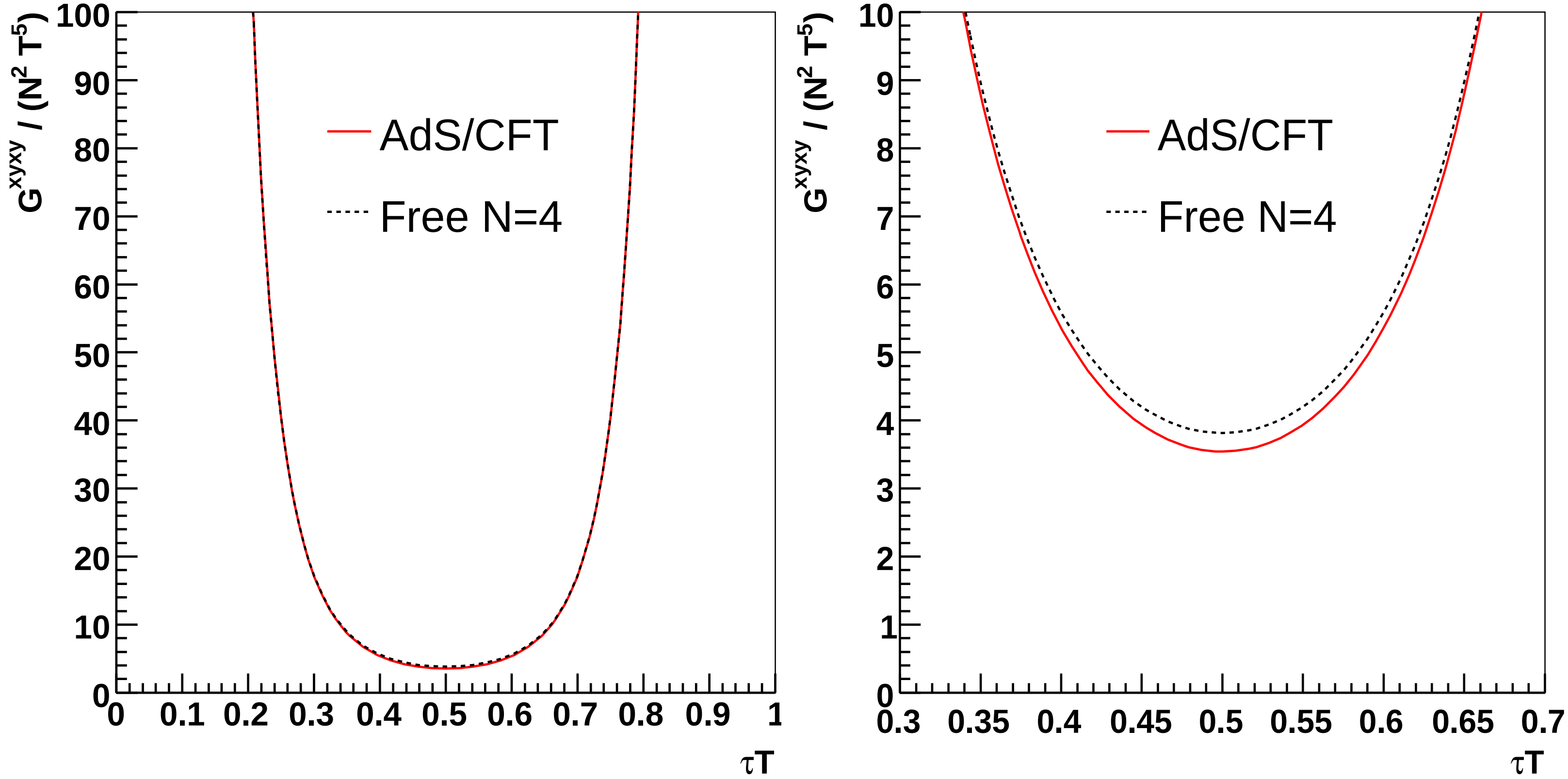
- A method to compute correlators of the stress tensor in $N = 4$ Super Yang Mills when $g^2 N \to \infty$.
- N=4 has 6 Scalars + 1 Guage Boson = 4 Left handed fermions
- Following strongly Son, Starinets, and Policastro.
  - They computed the shear viscosity, $\frac{\eta}{s} = \frac{1}{4\pi T}$
  - They left the spectral density for someone with a computer and interest.

N=4 Spectral Density



- Absolutely no hint of structure. No hint of a Debye scale of any kind
- The spectral density oscillates arround the zero temperature result with exponentially decreasing amplitude
- Lorentzian ansatz may be a poor choice.

Euclidean Correlator: Free and Strongly Interacting



- If you use perturbation theory and do a reasonable job on the pressure – You might trick yourself into thinking its true

Hydro Simulations

Model Equations (H.C. Ottinger 2001)

1. Imagine a tensor c_{ij} which relaxes quickly to $\partial_i v_j + \partial_j v_i$

$$\partial_t c_{ij} - (\partial_i v_j + \partial_j v_i) = \frac{\bar{c}_{ij}}{\tau_0} + \frac{\langle c_{ij} \rangle}{\tau_2}$$

where $\bar{c}_{ij} = (tr\,\mathbf{c})\,\delta_{ij}$ and $\langle c_{ij} \rangle = c_{ij} - \frac{1}{3}\bar{c}_{ij}$

- For small τ_0 and τ_2 we have:

$$c_{ij} \approx \tau_0 \delta_{ij}\, \partial_i v^i + \tau_2(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_l v^l)$$

2. Then the “effective” pressure for small strains is given by:

$$T_{ij} \approx p(\delta_{ij} - a_1\, c_{ij})$$

3. Compare this to the canonical form:

$$T_{ij} \approx p\delta_{ij} - \zeta \partial_i v^i - \eta(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_l v^l)$$

Can map, $(\tau_0, \tau_2, a_1) \rightarrow (\zeta, \eta, v_{\rm th}^2)$

Running Viscous Hydro in Three Steps

1. Run the evolution and monitor the viscous terms

2. When the viscous term is about half of the pressure:

   - The models disagree with each other.
   - T^{ij} is not asymptotic with $\sim \eta(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_l v^l)$

Freezeout is signaled by the equations.

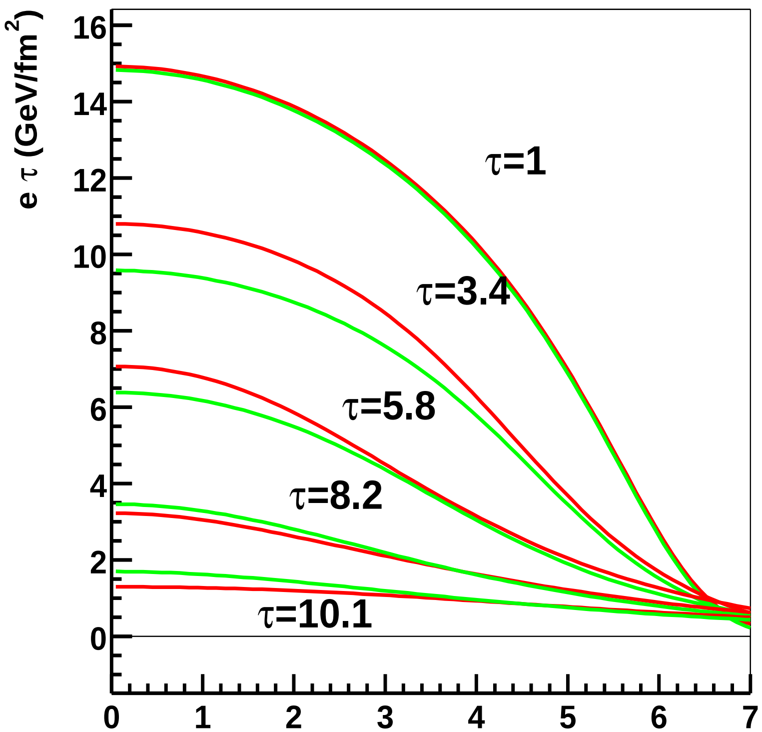
3. Compute spectra:

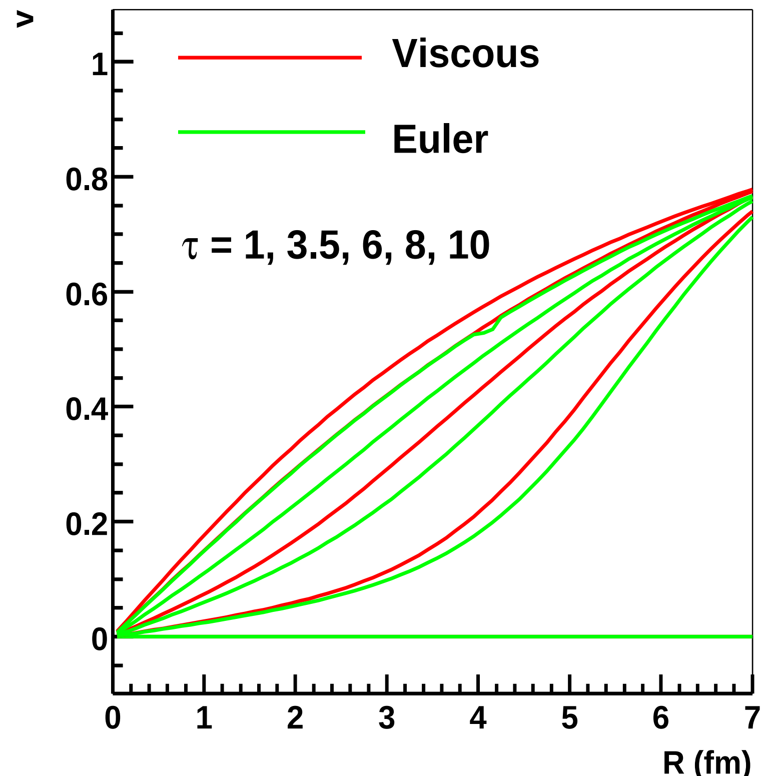
   - Viscous corrections to the spectra grow with p_T

$$f_o \longrightarrow f_o + \delta f$$

Maximum p_T is also signaled by the equations.

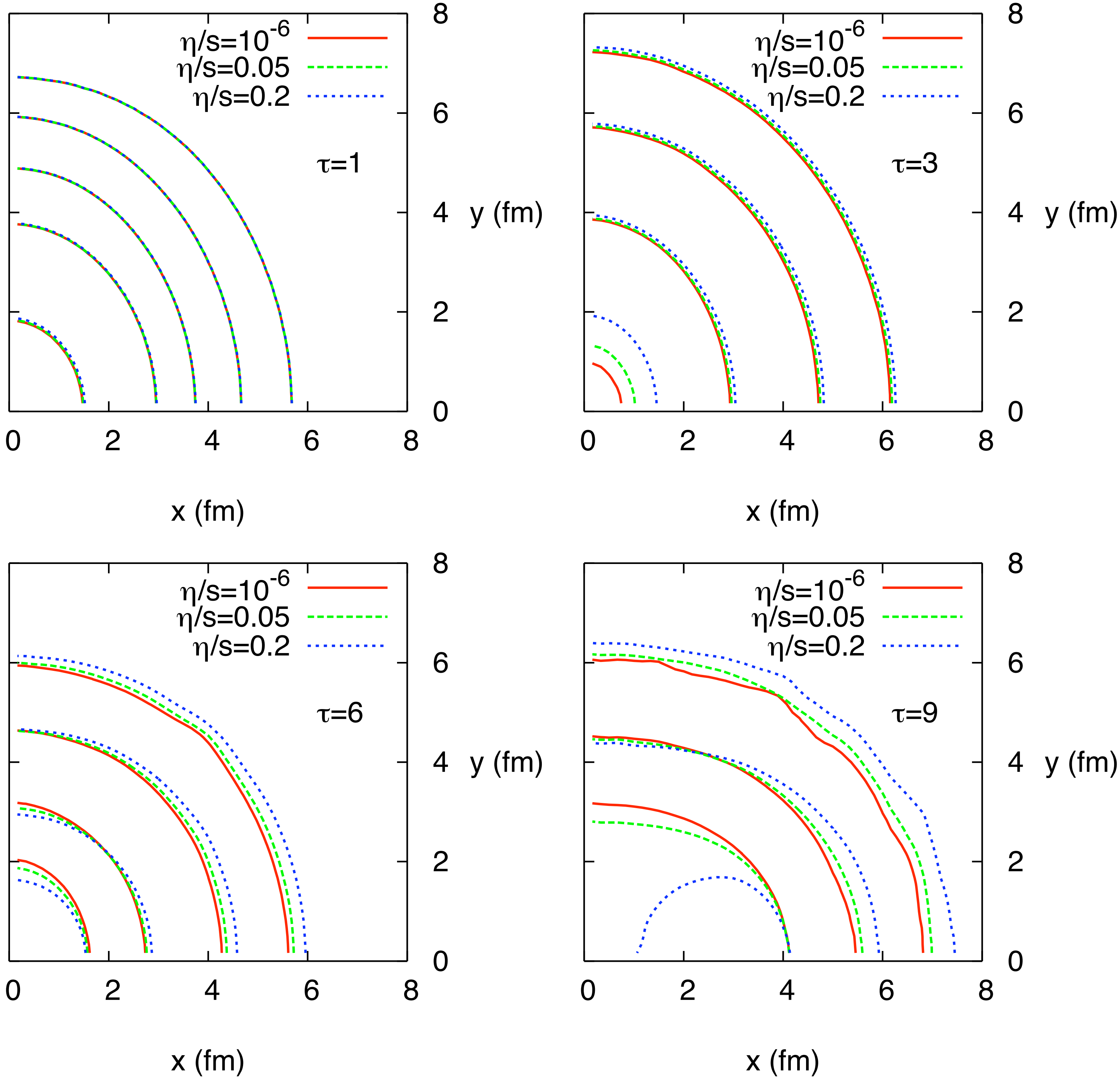
Bjorken Solution with transverse expansion: Step 1 ($\eta/s = 0.2$)





- First the viscous case does less longitudinal work.
- Then the transverse velocity grows more rapidly because the transverse pressure is larger.
- The larger transverse velocity then reduces the energy density more quickly than ideal hydro.

Viscous corrections do NOT integrate to give an O(1) change to the flow.



η/s=10^-6
η/s=0.05
η/s=0.2
τ=1
τ=3
τ=6
τ=9
x (fm)
y (fm)
0
2
4
6
8

Freezeout

- Freezeout when the expansion rate is too fast

$$\tau_R \partial_\mu u^\mu \sim 1$$

- The viscosity is related to the relaxation time

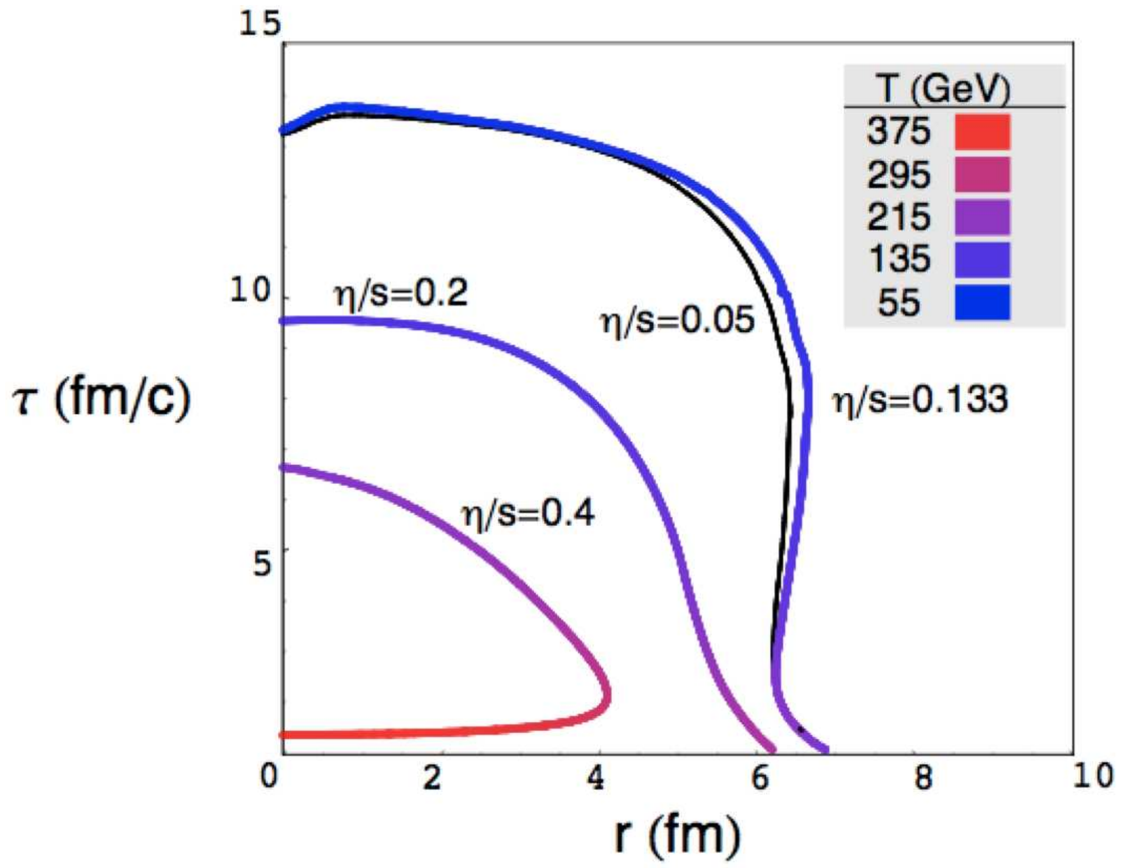
$$\frac{\eta}{e} \sim v_{\rm th}^2 \tau_R \qquad\qquad p \sim e\, v_{\rm th}^2$$

- So the freezeout cirterion is

$$\frac{\eta}{p}\,\partial_\mu u^\mu \sim 1$$

Monitor the viscous terms and compute freezeout: Step 2

- Contours where viscous terms become O(1)



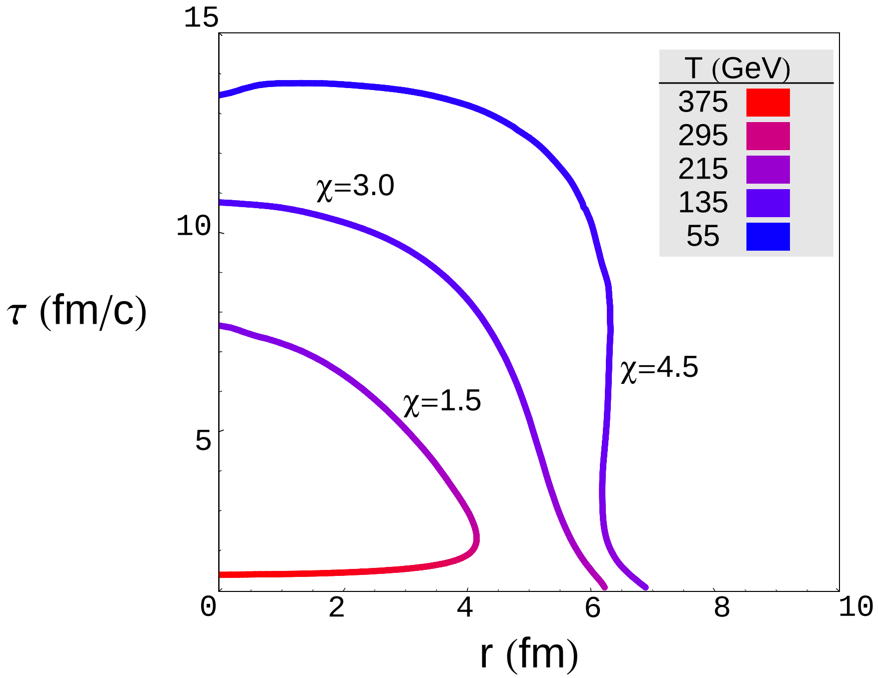
The space-time volume where hydro applies depends strongly on η/s

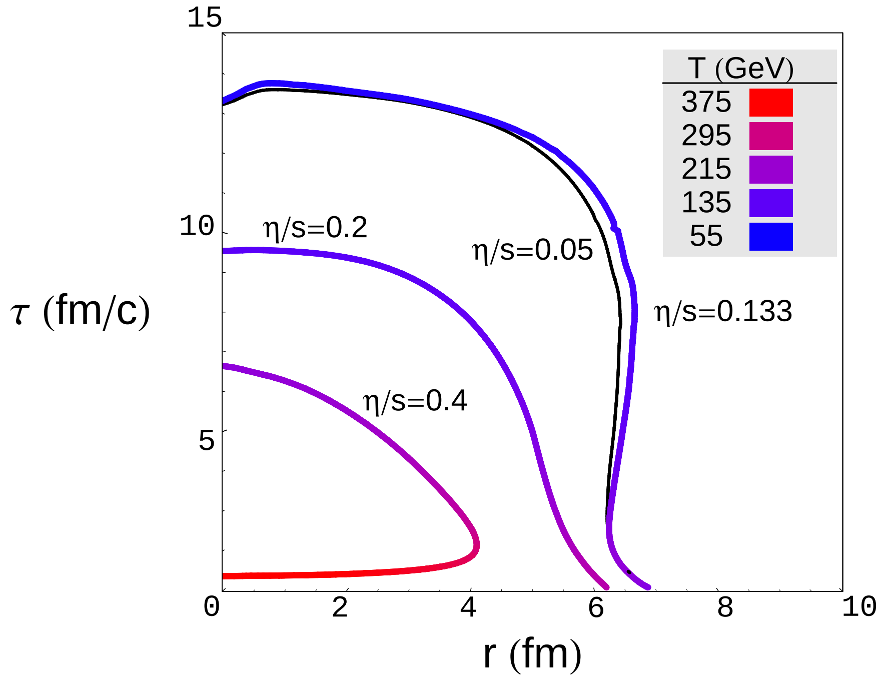
Decoupling Freezout and the Viscosity

- Freezeout at constant χ.

$$\frac{\eta}{p}\partial_\mu u^\mu = \frac{\eta}{s}\underbrace{\frac{4}{T}\partial_\mu u^\mu}_{\equiv\chi}$$

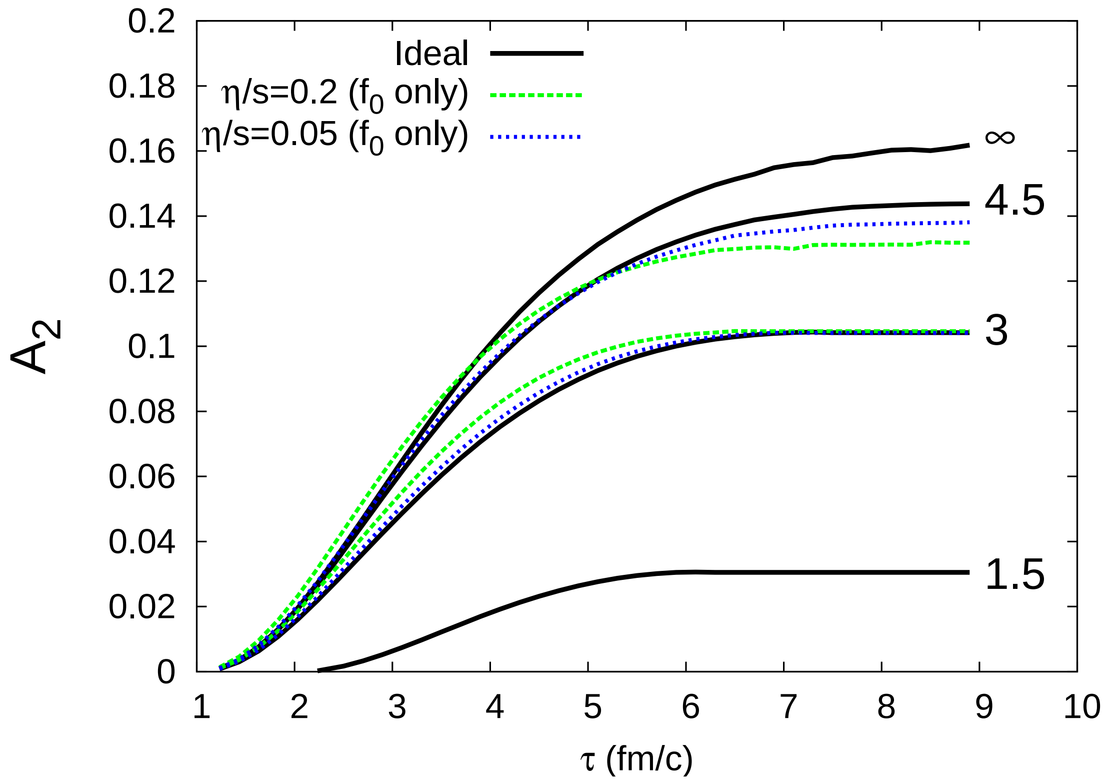
- The freezeout surface is independent of η/s also works for the ideal case





Elliptic Flow versus Time - No δf

$$\alpha_2 = \frac{\left\langle p_x^2 - p_y^2 \right\rangle}{\left\langle p_x^2 + p_y^2 \right\rangle} \approx 2\, v_2$$



Result without δf is insensitive to η/s (except through freezeout)

Elliptic Flow versus Time – with δf

- Corrections to thermal distribution function $f_0 \rightarrow f_0 + \delta f$
  - Must be proportional to strains
  - Must be a scalar
  - General form in rest frame and ansatz

$$\delta f = F(|\mathbf{p}|) p^i p^j \pi_{ij} \Longrightarrow \delta f \propto f_0 \, p^i p^j \pi_{ij}$$

  - Can fix the constant

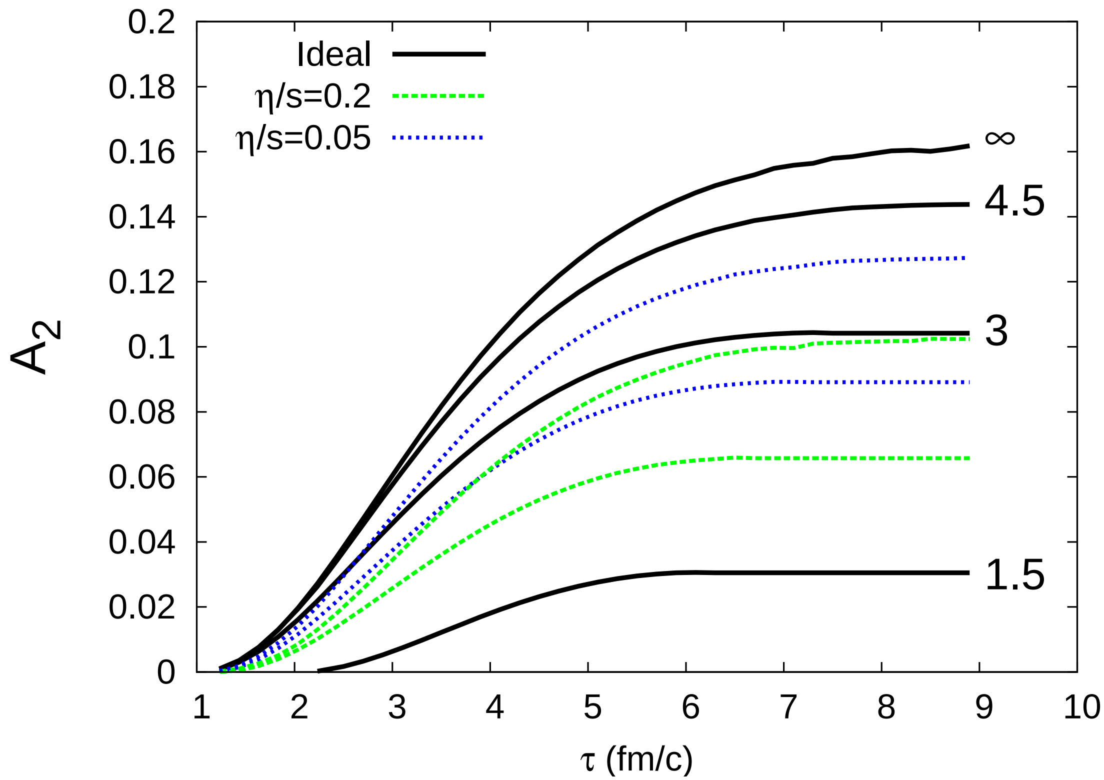
$$p\delta^{ij} + \pi^{ij} = \int \frac{d^3p}{(2\pi)^3} \frac{p^i p^j}{E_{\mathbf{p}}} (f_0 + \delta f)$$

find

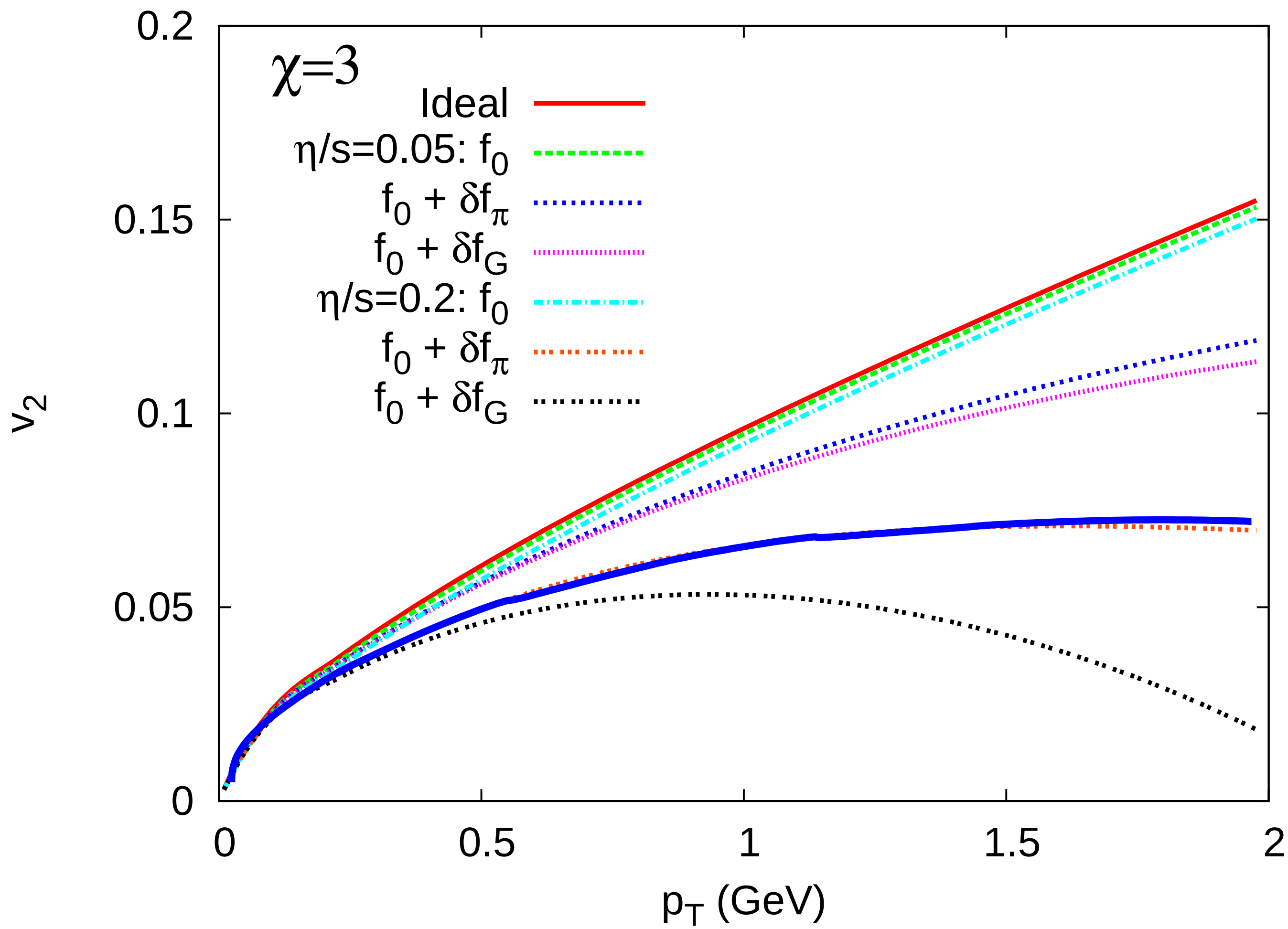
$$\delta f = \frac{1}{2(e+p)T^2} f_o \, p^i p^j \pi_{ij}$$

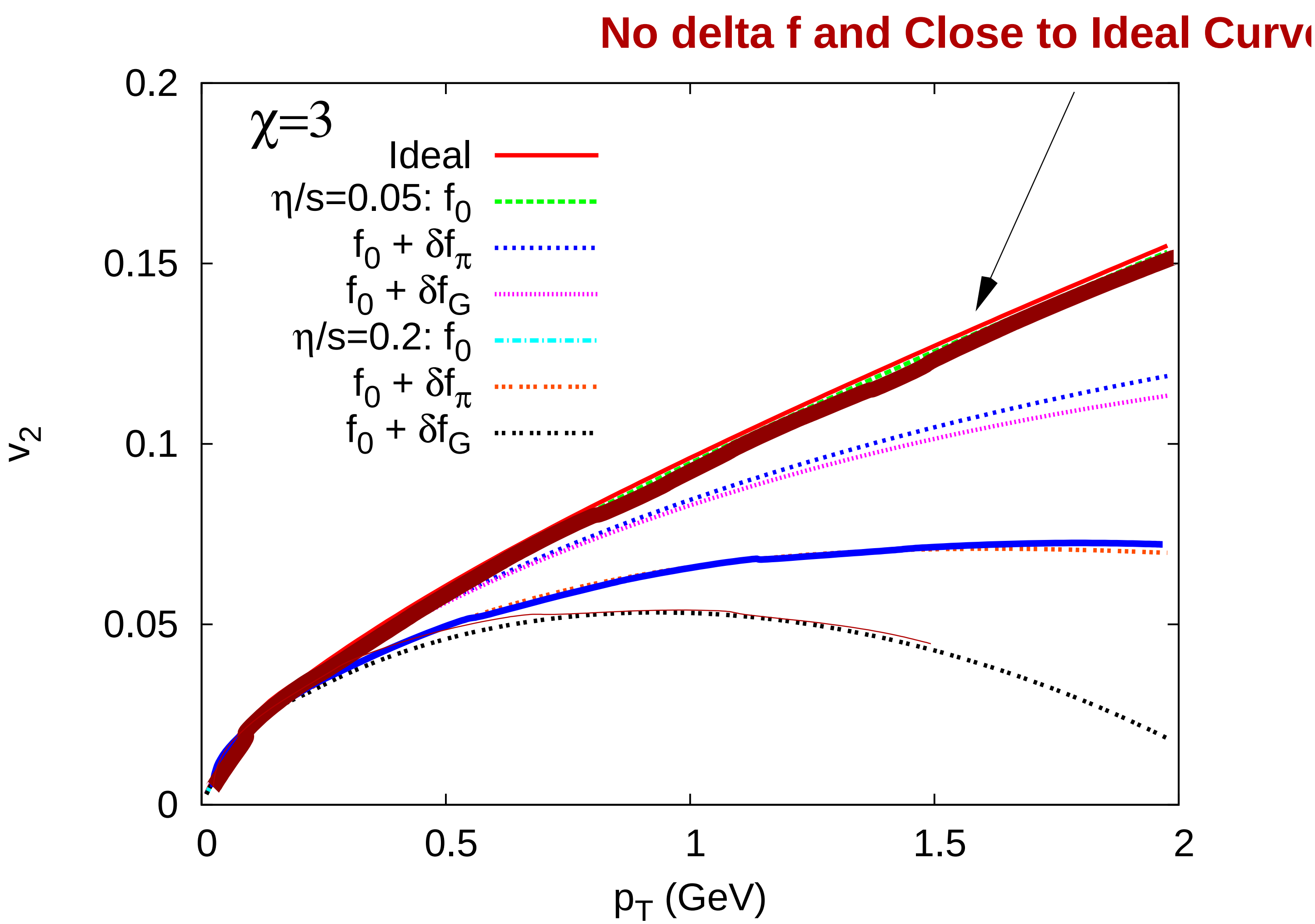
Elliptic Flow versus Time - with δf

$$\alpha_2 = \frac{\left\langle p_x^2 - p_y^2 \right\rangle}{\left\langle p_x^2 + p_y^2 \right\rangle} \approx 2\, v_2$$



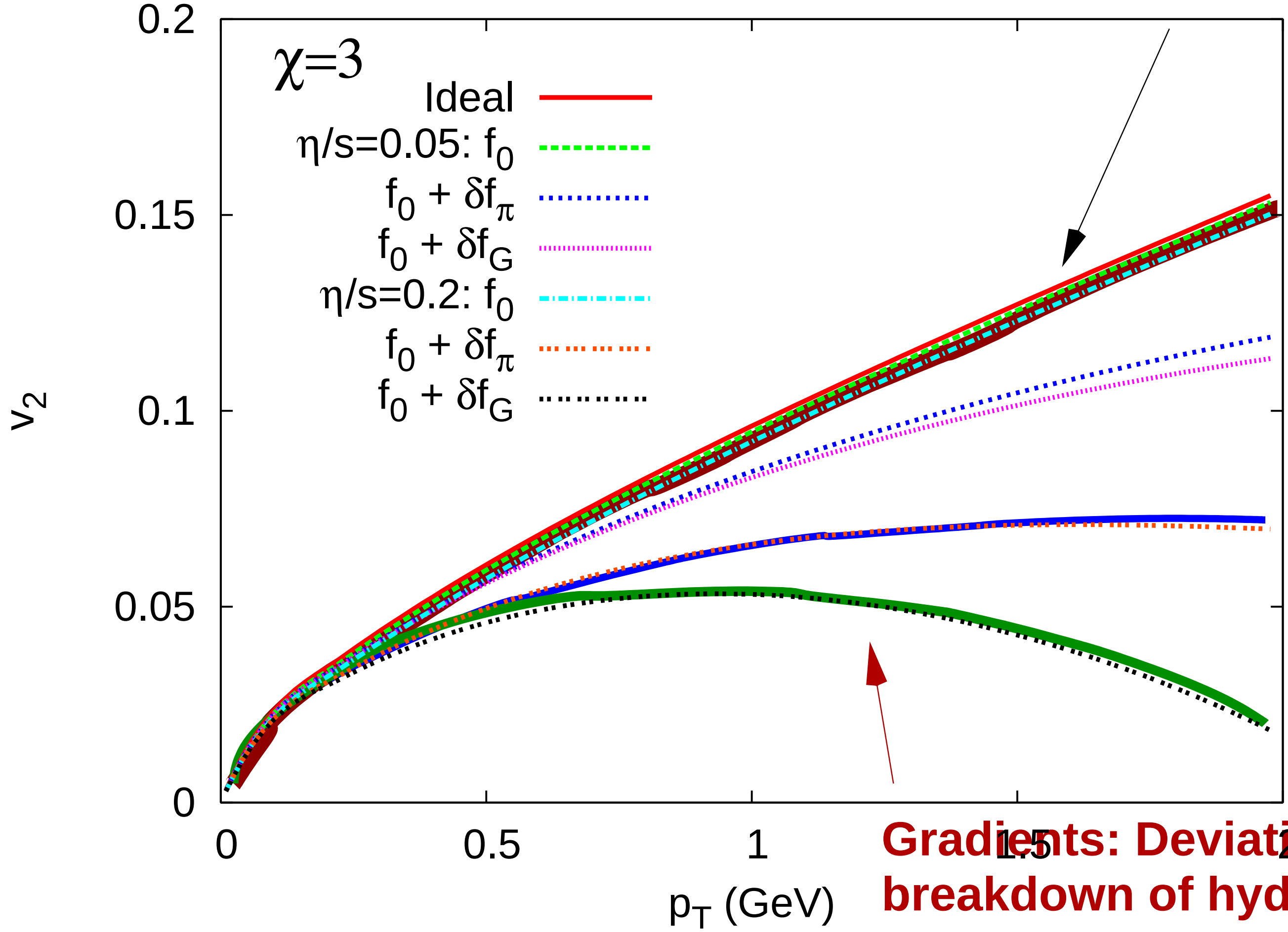
Elliptic Flow as a function of viscosity and p_T, $\eta/s = 0.2$





No delta f and Close to Ideal Curve
χ=3
Ideal
η/s=0.05: f0
f0 + δfπ
f0 + δfG
η/s=0.2: f0
f0 + δfπ
f0 + δfG
v2
0
0.05
0.1
0.15
0.2
0
0.5
1
1.5
2
pT (GeV)

No delta f and Close to Ideal Curve

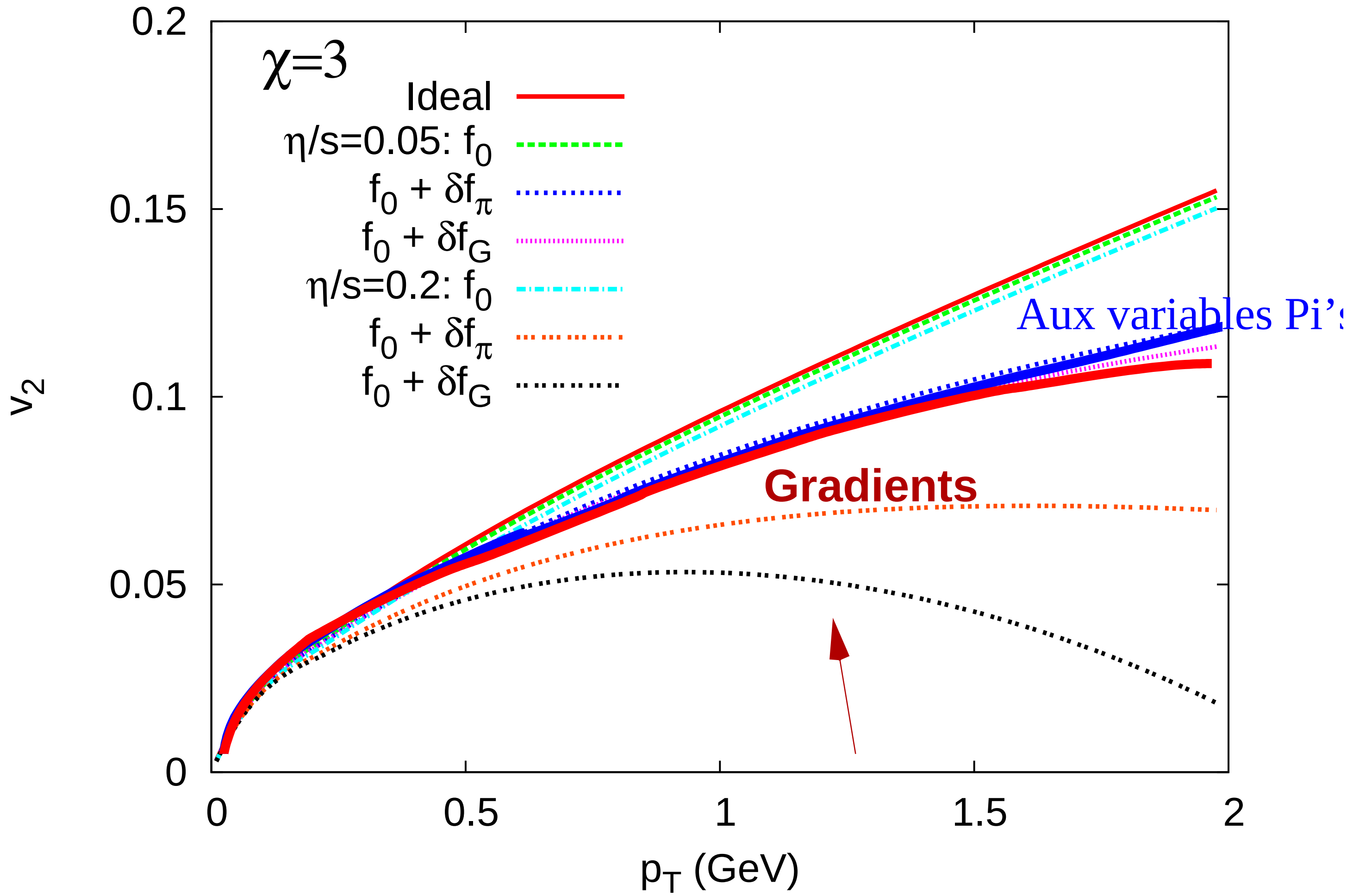


Gradients: Deviations signa
breakdown of hydro

$$\eta \left\langle \partial^i v^j \right\rangle = \eta \left(\partial^i u^j + \partial^j u^i - \frac{2}{3} \partial_l u^l \delta^{ij} \right)$$

Estimates the uncertainty

Compare to $\eta/s = 0.05$



$\eta \left\langle \partial^i v^j \right\rangle$ and π^{ij}

Conclusions:

- Viscosity does not change the ideal hydrodynamic solution much. Time is not very long.
- Viscosity signals the boundary of applicability of hydro
  - Need $\eta/s < 0.3$ in order that hydro describe a significant fraction of the collision space-time volume
- In order to obatin $v_2^{\rm vis} \approx \frac{2}{3}\, v_2^{\rm ideal}$ need $\eta/s < 1/6$
- Large ambiguities for $\eta/s > 0.3$