ENTROPY AND VISCOSITY

IN STRONGLY COUPLED

$\mathcal{N}=4$ SYM THEORY

FROM ADS/CFT DUALITY

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Based on
Gubser, IK, Peet, hep-th/9602135
etc.
reviewed in I.K., hep-th/9901018
Policastro, Son, Starinets,
hep-th/0104066

DURING THE LAST SEVERAL YEARS NEW INSIGHTS INTO GAUGE THEORY HAVE BEEN OBTAINED BY EMBEDDING IT INTO STRING THEORY IN A NOVEL WAY.

Gauge theories can be made to live on multi-dimensional extended objects called D-branes.

These gauge theories are typically supersymmetric relatives of SU(N) gauge theory discussed above.

The simplest example is to study 3-dimensional D-branes (the 3-branes) embedded into 9+1 dimensional superstring theory.

The gauge theory on 3-branes is the maximally supersymmetric 3+1 dimensional gauge theory.

In addition to gluons it contains 6 scalar fields and 4 fermions. Now, the coupling g_{YM} does not "flow" => the theory is CONFORMAL

D-branes: where strings may end, even in

type II theories, where away from the D-branes there are closed strings only.

Dai, Leigh, Polchinski; Polchinski

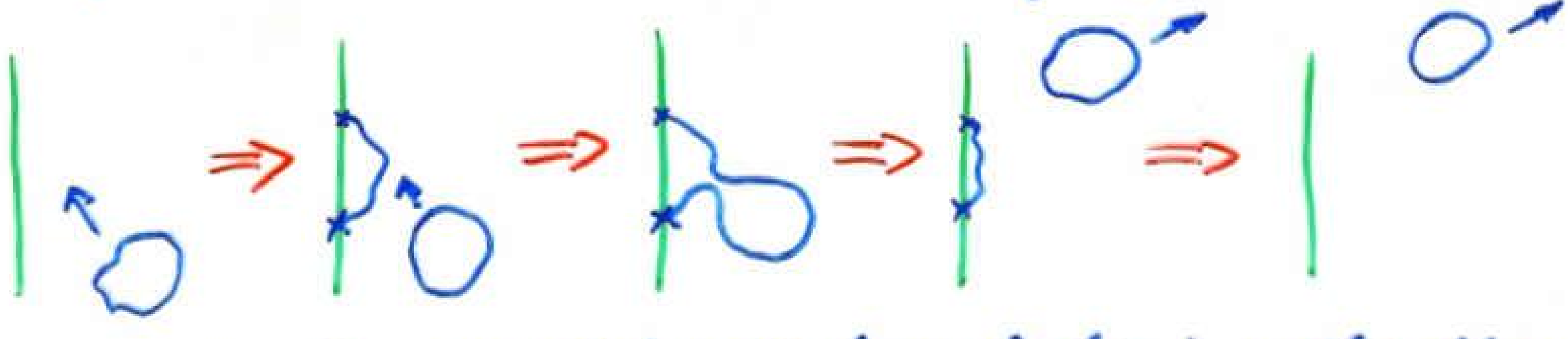
There are $p+1$ longitudinal (world volume) coordinates, with Neumann boundary conditions at the end-points: $n^a \partial_a X^A = 0$; $A = 0, \ldots, p$.

$9-p$ transverse coordinates with Dirichlet boundary conditions: $X^i = X^i_0$; $i = p+1, \ldots, 9$.

X^i_0 are the coordinates of the D-brane.

The open strings make the D-brane dynamical.

For example, they mediate interactions with closed strings. Consider $1 \to 1$ scattering off D-branes



Leading order gravitational form-factor is easily obtained

Type II SUGRA admits R-R charged p-brane solutions (Horowitz and Strominger; Duff and Lu):

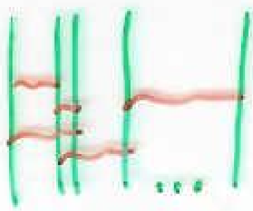
$$ds^2 = H^{-\frac{1}{2}}(-dt^2 + dx_1^2 + \ldots + dx_p^2) + H^{\frac{1}{2}}(dr^2 + r^2 d\Omega_{8-p}^2)$$

$$e^{\Phi} = H^{\frac{3-p}{4}}; \qquad H(r) = 1 + \frac{R^{7-p}}{r^{7-p}},$$

where the metric is given in the string frame.

These solutions are extremal: they preserve 16 of the original 32 supersymmetries.

Due to the magic of D-branes, we know a different (dual) description of the p-brane in terms of N parallel Dp-branes.



Here we find a $U(N)$ gauge theory with 16 supercharges (Witten)

The $9-p$ scalar expectation values, $\langle X^i \rangle$ label the relative transverse positions of the Dp-branes. To describe the single center object we take $\langle X^i \rangle = 0$: all N Dp-branes are coincident.

For an appropriate choice of the parameters (N, g_{str}, α') the N coincident D-branes carry the same charge under the R-R $(p+1)$-form gauge potential (Polchinski) and the same SUSY.

Hence, they should be regarded as different descriptions of the same object in type II string theory.

All SUGRA p-branes are singular, due to divergence of Φ at $r=0$, except for $p=3$.

Thus, we focus on the black 3-brane as a dual description of the $D=4$ $SU(N)$ gauge theory with 16 supercharges.

We equate the ADM tension to N times that of a D3-brane to obtain (Gubser, IK, Peet)

$$\frac{2\pi^3 R^4}{\kappa^2} = \frac{\sqrt{\pi}}{\kappa} N \Rightarrow R^4 = \frac{\kappa N}{2\pi^{5/2}} \sim g_{str} N (\alpha')^2 = 2 g_{YM}^2 N (\alpha')^2.$$

There are 2 sources of α' corrections in studying string physics on the background of a black 3-brane.

Corrections due to the finite radius of the throat are $\sim \frac{\alpha'}{R^2} \sim \frac{1}{\sqrt{g_{st} N}}$;

For processes with typical energy scale ω there are also corrections in powers of $\omega^2 \alpha'$.

To suppress both types of string corrections to supergravity, we need to take the double-scaling limit (IK):

$$N g_{st} \to \infty \quad ; \quad \omega^2 \alpha' \to 0.$$

In this limit SUGRA gives exact information about the $D=4$ $SU(N)$ SYM theory!

String loop corrections are suppressed if $g_{str} \to 0$.

Thus, we also need to take $N \to \infty$.

Since $2\pi g_{str} = g_{YM}^2$, the black 3-brane gives us PREDICTIONS concerning the strong 't Hooft coupling ($g_{YM}^2 N$) behavior of the large N SYM theory.

If $g_{YM}^2 N$ is kept fixed but very large, then the string α' corrections may be used to develop an expansion in $(g_{YM}^2 N)^{-\frac{1}{2}}$

The string loop corrections are suppressed by positive powers of $\frac{\kappa^2}{R^8} \sim \frac{1}{N^2}$. The large N limit is mapped onto classical string physics.

One application of these principles: by studying absorption of low-energy particles incident on the black 3-brane, we can deduce exact 2-point correlators of the strongly coupled large-N SYM theory.

Consider, for instance, absorption of a dilaton (IK) whose propagation is governed by $\Box \Phi = 0$.

For s-waves (no dependence on S^5), setting $\rho = \omega r$,

$$\left[\rho^{-5}\frac{d}{d\rho}\rho^5\frac{d}{d\rho} + 1 + \frac{(\omega R)^4}{\rho^4}\right]\Phi(\rho) = 0.$$

For $\rho \ll 1$ (in the throat region) the solution is

$$\Phi_{throat} = i\,(\omega R)^4 \rho^{-2} H_2\left(\frac{(\omega R)^2}{\rho}\right).$$

For $\rho \gg (\omega R)^2$ (in the asymptotic region)

$$\Phi_{asymp} = \frac{32}{\pi}\rho^{-2} J_2(\rho).$$

The solutions are easily matched for $(\omega R) \ll 1$.

The low-energy limit of the absorption probability is $P = \frac{\pi^2}{16^2}(\omega R)^8$. More sophisticated methods, which give an exact function of ωR were recently used by Gubser and Hashimoto.

From P we find the low-energy absorption cross-section $\sigma = \frac{\pi^4}{8}\omega^3 R^8$, which can be used for comparison with the SYM theory.

Taking the low-energy limit of the Born-Infeld action, which describes N D3-branes in external SUGRA fields, we find the bosonic part of the action,

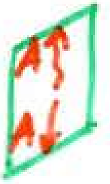
$$S = T_{(3)}\int d^4\sigma\, \mathrm{Tr}\left[-\frac{1}{2}\partial_\alpha X^i \partial^\alpha X^i - \frac{1}{4}e^{-\phi}F_{\alpha\beta}^2 - C F_{\alpha\beta}\tilde{F}^{\alpha\beta} + \dots\right]$$

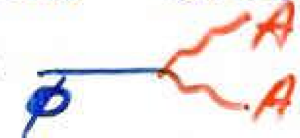
where C is the R-R scalar.

The bulk action is

$$S_{bulk} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left(R - \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}e^{2\phi}(\partial_\mu C)^2 + \dots\right)$$

This establishes a correspondence between the fields of SUGRA and gauge invariant operators of SYM theory: $\phi \longleftrightarrow \mathrm{Tr}\,F^2$, $C \longleftrightarrow \mathrm{Tr}\,F\tilde{F}$, ...

To lowest order in g_{st}, the dilaton absorption is due to 

$\xleftarrow{\ \phi\ }$. The matrix element comes from the diagram: $M =$ 

$$\sigma = \frac{1}{2} N^2 \frac{1}{2\omega} \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \delta(E_1+E_2-\omega)\, \delta^3(\vec{p}_1+\vec{p}_2) \sum |\bar{M}|^2 .$$

symmetry factor

$U(N)$ factor

$$\sum |\bar{M}|^2 = |\bar{M}|^2_{\text{gauge bosons}} = \kappa^2 \omega^4 .$$

Thus, $\sigma = \dfrac{\kappa^2 N^2 \omega^3}{32\pi} = \dfrac{\pi^4}{8} \omega^3 R^8$,

in exact agreement with the semiclassical SUGRA!

For the dilaton, pairs of scalars and fermions do not contribute to absorption.

Gravitons polarized along the 3-brane, $h_{\alpha\beta}$, are also minimally coupled scalars from the $D=7$ point of view.

Their world volume interaction is

$$\delta S_{(3)} = T_{(3)} \int d^4\sigma \, \frac{1}{2} h_{\alpha\beta} T^{\alpha\beta} ,$$

$$T_{\alpha\beta} = \mathrm{Tr}\Big[F_{\alpha\gamma} F_\beta{}^\gamma - \frac{1}{4}\eta_{\alpha\beta} F^2 - \frac{i}{2} \bar{\psi}^I \gamma_{(\alpha} \partial_{\beta)} \psi_I + \partial_\alpha X^i \partial_\beta X^i - \frac{1}{2}\eta_{\alpha\beta} (\partial_\gamma X^i)^2 \Big],$$

the world volume stress-energy tensor.

For final particle momentum along $\hat{n}$,

$$|\bar{M}|^2_{\text{scalars}} = \kappa^2 \omega^4 \times 3 n_x^2 n_y^2 \; ; \; |\bar{M}|^2_{\text{fermions}} = \kappa^2 \omega^4 (n_x^2 + n_y^2 - 4 n_x^2 n_y^2)$$

$$|\bar{M}|^2_{\text{gauge bosons}} = \kappa^2 \omega^4 (1 - n_x^2 - n_y^2 + n_x^2 n_y^2)$$

are the sums over final polarizations for incident h_{xy}.

$\sum |\bar{M}|^2 = \kappa^2 \omega^4 \Longrightarrow$ the graviton absorption cross-section is the same as for dilaton, in agreement with SUGRA!

So far, our calculations are equivalent to finding the discontinuity (imaginary part) on the 1-loop diagrams

;

; etc.

Comparison with gravity allows to deduce the $g^2_{YM} N \to \infty$ limit of the exact 2-point functions.

$$\langle \mathrm{tr}\,F^2(x)\ \mathrm{tr}\,F^2(0) \rangle = k\,\frac{N^2}{|x|^8} ;$$

$$\langle T_{\alpha\beta}(x)\ T_{\gamma\delta}(0) \rangle = \frac{N^2/4}{48\pi^4}\, X_{\alpha\beta\gamma\delta}\left(\frac{1}{x^4}\right)\tilde{k} ;$$

We find that $\tilde{k} = 1$ both for $g^2_{YM} N = 0$ and for $g^2_{YM} N \to \infty$; $k(g^2_{YM} N = 0) = k(g^2_{YM} N = \infty)$.

This is consistent with the non-renormalization of the central charge: it is related to the Adler-Bardeen theorem for the R-current anomaly by $N = 4$ SUSY.

FINITE TEMPERATURE THEORIES

are among the nicest examples of how gravity provides dynamical information about strongly coupled gauge theory.

Consider a stack of N D3-branes heated up to temperature T.

The corresponding metric is known exactly and has the Hawking temperature T.

In gravitational systems one associates the Bekenstein-Hawking entropy

$$S_{BH} = \frac{A}{4G}$$

to event horizon of area A.

For the metric describing thermal 3-branes it was found (Gubser, Peet, IK)

$$S_{BH} = \frac{\pi^2}{2} N^2 T^3 V .$$

This is the thermal SYM entropy in the $g_{YM}^2 N \to \infty$ limit, according to the duality

$$ds^2 = h^{-\frac{1}{2}}(r)\left[-f(r)\,dt^2 + \sum_{i=1}^{3}(dx^i)^2\right] + h^{\frac{1}{2}}(r)\left[\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right],$$

$$h(r) = 1 + \frac{L^4}{r^4}; \quad f(r) = 1 - \frac{r_0^4}{r^4}.$$

Now the horizon is located at $r = r_0 \ll L$.

The near-horizon geometry ($r \ll L$) is

$$ds^2 \to \frac{r^2}{L^2}\left[-\left(1 - \frac{r_0^4}{r^4}\right)dt^2 + \sum_{i=1}^{3}(dx^i)^2\right] + \frac{L^2}{r^2}\frac{dr^2}{1 - \frac{r_0^4}{r^4}} + L^2 d\Omega_5^2.$$

This is a particular example of a black hole in AdS_5.

Its Euclidean continuation is asymptotic to $S^1 \times R^3$.

The circumference of S^1 is $\beta = \frac{1}{T}$.

β is determined by the absence of a conical singularity at $r = r_0$.

Define ρ through $r = r_0 + \frac{r_0}{L^2}\rho^2$; $\tau = it$.

The $\rho\tau$-metric is

$$d\rho^2 + \frac{4r_0^2}{L^4}\rho^2 d\tau^2 = d\rho^2 + \rho^2 d\theta^2; \quad \theta \in [0, 2\pi)$$

The circumference of the S^1 is $\beta = \frac{\pi L^2}{r_0}$.

$$T = \frac{1}{\beta} = \frac{r_0}{\pi L^2} \ll \frac{1}{L}.$$

8-d horizon area $A = \frac{r_0^3}{L^3} V L^5 \pi^3 =$

$$= \pi^6 L^8 V T^3.$$

Since $L^8 = \frac{N^2}{4\pi^5} K^2$,

$$\boxed{S_{BH} = \frac{2\pi A}{K^2} = \frac{\pi^2}{2} N^2 T^3 V}.$$

T^3 scaling as expected in a CFT;

N^2 scaling characteristic of $SU(N)$ gauge theory.

On general field theory grounds we have

$$S = f(g_{YM}^2 N)\,\frac{2\pi^2}{3} N^2 T^3 V,$$

power of T fixed by scale invariance

and perturbative calculations give

$$f = 1 - \frac{3}{2\pi^2} g_{YM}^2 N + \frac{3+\sqrt{2}}{\pi^3} (g_{YM}^2 N)^{3/2} + \dots$$

Gravity results at strong coupling show that

$$f = \frac{3}{4} + \frac{45}{32} \zeta(3) (2 g_{YM}^2 N)^{-3/2} + \dots$$

Thus, $f(g_{YM}^2 N)$ is probably a smooth function interpolating between 1 and $3/4$

SHEAR VISCOSITY

(Policastro, Son, Starinets)

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt\, dx\, e^{i\omega t} \langle [T_{xy}(t,\vec{x}), T_{xy}(0,0)] \rangle$$

At weak coupling $g^2 N \ll 1$,

$$\eta \sim \frac{N^2 T^3}{(g^2 N)^2 \ln(1/g^2 N)};$$

$$\frac{\eta}{s} \sim \frac{1}{(g^2 N)^2 \ln(1/g^2 N)}$$

Diverges as $g^2 N \to 0$.

To calculate at strong coupling, use absorption by near-extremal 3-branes:

$$\sigma(\omega) = \frac{\kappa^2}{\omega} \int dt\, dx\, e^{i\omega t} \langle [T_{xy}(t,\vec{x}), T_{xy}(0,0)] \rangle$$

for gravitons h_{xy}.

In general, $\sigma(0)$ = Horizon Area

For 3-branes,

$$\sigma(0) = \pi^3 r_0^3 R^2$$

The viscosity at $g^2 N \to \infty$,

$$\eta = \frac{1}{2\kappa^2}\sigma(0) = \frac{\pi}{8} N^2 T^3,$$

where again we used $T = \frac{r_0}{\pi R^2}$.

Since $s = \frac{S}{V} = \frac{\pi^2}{2} N^2 T^3$,

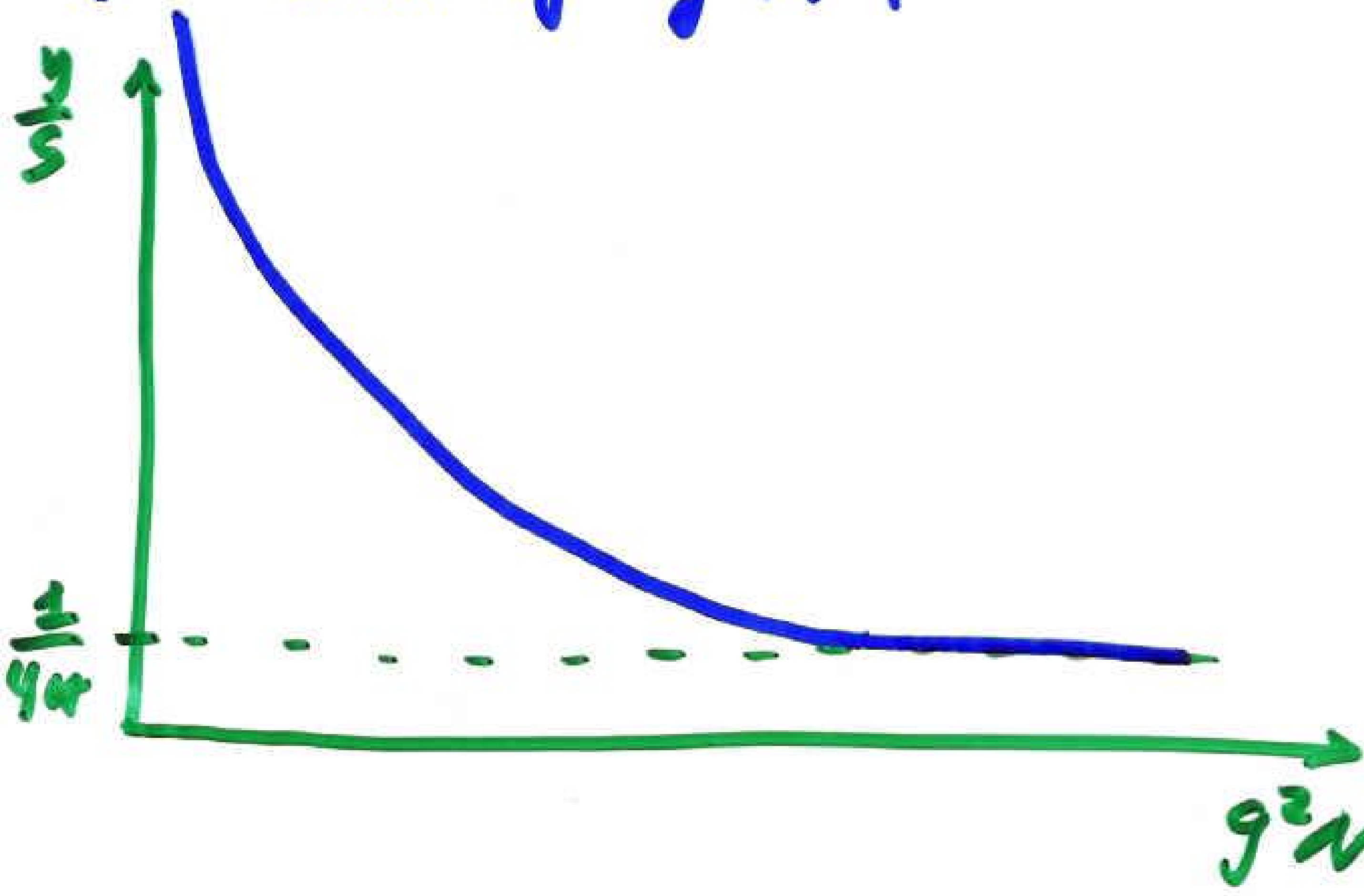
$$\frac{\eta}{s} = \frac{1}{4\pi} \quad \text{for } g^2 N \to \infty.$$

Is this the lowest possible value?
(Kovtun, Son, Starinets)

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B} \approx 6.08 \times 10^{-13}\ \mathrm{K\text{-}s}$$

For water under normal conditions,
$\eta/s \approx 2.3 \times 10^{-10}$ K-s (400 larger than $\frac{\hbar}{4\pi k_B}$)
Liquid ^{4}He at 1 MPa, $T = 10$ K $\Rightarrow \approx 6 \times 10^{-12}$ K-s.

It is plausible (though not yet checked), that $\frac{\eta}{s}$ is a monotonic function of g^2N:



Leading correction for large g^2N, $\mathcal{O}((g^2N)^{-3/2})$ comes from the R^4 correction to supergravity.