Small-x and gluon saturation

(Lecture notes of Prof. Bowen Xiao at Huada school on QCD 2016)

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Small-x and gluon saturation

- DIS
- DGLAP eq.
- Small-x evolution eq. (BFKL , BK , ⋯)
- Resummation

Reference: quantum chromodynamics at high energy, Yuriv. Kovchegov & Eugene Levin
(details)

notations and conventions:

normal metric tensor : $g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$x^\mu = (x^0, \vec{x})$ $\quad p^\mu = (p^0, \vec{p})$

$x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$ $\quad p_\mu = g_{\mu\nu} p^\mu = (p^0, -\vec{p})$

$p \cdot x = g_{\mu\nu} p^\mu x^\nu = p^0 x^0 - \vec{p} \cdot \vec{x}$

light cone coordinates:

two definitions: For an arbitrary 4-vector $V^\mu = (V^0, V^1, V^2, V^3)$

✓ ① $V^+ \equiv V^0 + V^3$, $V^- \equiv V^0 - V^3$, $V_\perp \equiv (V^1, V^2)$. → note: Yuriv. Kovchegov's book use this convention.

(we use ① in the following lectures)

$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} & & \\ \frac{1}{2} & 0 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ but $g^{\mu\nu} \neq g_{\mu\nu}$

$U \cdot V = \frac{1}{2}(U^+V^- + U^-V^+) - U_\perp \cdot V_\perp$

$V^2 = (V^0)^2 - V_\perp^2 - (V^3)^2 = V^+V^- - V_\perp^2 = \frac{1}{2}V^+V^- + \frac{1}{2}V^-V^+ - V_\perp^2$

② $V^+ \equiv \frac{1}{\sqrt{2}}(V^0 + V^3)$, $V^- \equiv \frac{1}{\sqrt{2}}(V^0 - V^3)$, $V_\perp \equiv (V^1, V^2)$ → note: literature often use this convention.

$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$V^2 = 2V^+V^- - V_\perp^2$

$U \cdot V = U^+V^- + U^-V^+ - U_\perp V_\perp$

In particular, we shall refer to

$$x^+ = \underset{\downarrow \atop t}{x^0} + \underset{\downarrow \atop z}{x^3} \rightarrow \text{light cone time}$$

$$x^- = x^0 - x^3 \rightarrow \text{light cone "longitudinal coordinate".}$$

p^2 for onshell particle $p^2 = m^2 = p^+p^- - p_\perp^2$

p^+ always dot x^-

p^- always dot x^+

$$p^\mu = (p^+, \; p^- = \frac{p_\perp^2 + m^2}{p^+}, \; p_\perp)$$

on-shell particle

in LC coordinate: right-mover $p^+ \gg p^-$

relativistic left-mover $p^- \gg p^+$

rapidity: in LC coordinate

$$y \equiv \frac{1}{2} \ln \frac{p^+}{p^-}$$ (in high energy limit, pseudorapidity is equal to rapidity) $\Rightarrow p^+ = \frac{1}{\sqrt{2}} m_\perp e^y, \; p^- = \frac{1}{\sqrt{2}} m_\perp e^{-y}$

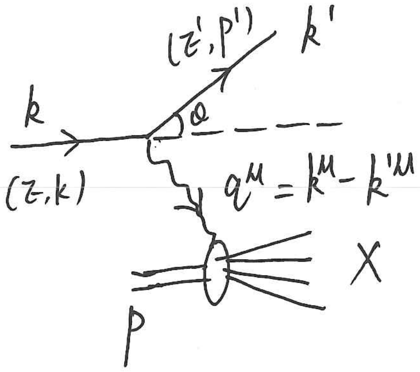
$p_0^2 - p_3^2 = p_\perp^2 + m^2 \equiv m_\perp^2 \Rightarrow m_\perp^2 \cosh^2 y - m_\perp^2 \sinh^2 y = m_\perp^2$

another way to define rapidity: $p_0 = m_\perp \cosh y$, $p_3 = m_\perp \sinh y$

In lorentz boost along z axis, $p^\mu \rightarrow p^{\mu\prime}$, just change rapidity: by p^+, p^-

$$\begin{cases} p_\perp' = p_\perp \\ p_0' = \gamma(p_0 - \beta p_z) \\ p_z' = \gamma(p_z - \beta p_0) \end{cases} \Rightarrow \begin{cases} p^{+\prime} = k p^+ \\ p^{-\prime} = k^{-1} p^- \end{cases} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \text{where } k = \sqrt{\frac{1-\beta}{1+\beta}}$$

DIS:



inclusive DIS

in target frame $=\!=\!=$ $(q^0, q_1, 0)$ → photon is transverse.

4 kinds of frames to see DIS

1. target rest frame
2. Bjorken frame
3. Breit frame
4. Dipole frame

Breit frame:

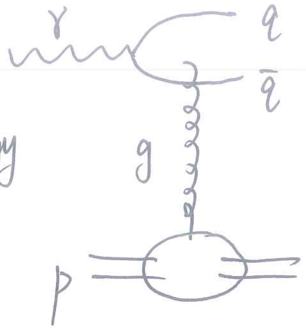
(Brick-wall frame)

$\vec{p}' = -\vec{p}$ $q^\mu = (\overset{0}{0}, \overset{1}{0}, \overset{2}{0}, \overset{3}{+q_3})$

dipole frame:

useful in small x and dijet production!

high energy & small-x



$$q^\mu_{dipo} = (\overset{q_0+q_3}{\overset{\uparrow}{q^+}}, \frac{-Q^2}{q^+}, q_\perp = 0)$$

use light-cone variable

note: Different frame has different physical picture, but physics is the same.

$e + p \to e + X$: some variables are Lorentz invariant~~s~~ quantities, but some are not Lorentz invariant quantities.

define invariants:

(1) $q^2 = (k-k')^2 = -2k\cdot k' = -2EE'(1-\cos\theta) \equiv -Q^2$ → always positive

(neglect the mass of electron)

(2) $P^\mu = (M, \vec{0})$ $P^\mu = (P^+, \frac{M^2}{P^+}, 0_\perp)$

(in target rest frame)

(3) $y \equiv \frac{P\cdot q}{P\cdot k} \underset{\text{in target rest frame}}{=} \frac{M(E-E')}{ME} = 1 - \frac{E'}{E}$

sometimes define $\nu \equiv E - E'$, $\nu = yE$ in target rest frame.

(4) $x_{Bj} \equiv \frac{Q^2}{2P\cdot q}$ particularly in parton model.

Parton model have two assumptions:

① Partons in a fast-moving hadron are moving collinear as the parent hadron.

② Reaction rate can be viewed as the incoherent sum of partonic scattering X-sections.

here refer to e+p scattering

$\Sigma|M|^2$

free parton

certain time-scale:

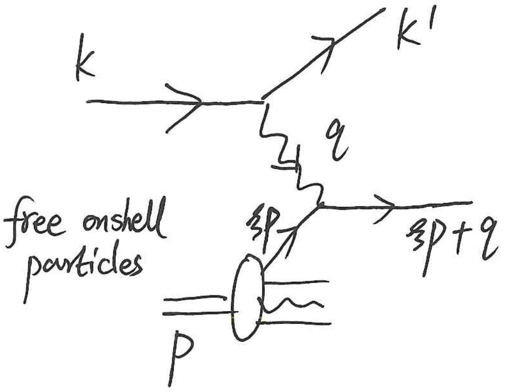
$$x_{Bj} \equiv \frac{Q^2}{2P\cdot q} \overset{\text{in target rest frame}}{=\!=\!=\!=} \frac{Q^2}{2Mq^0} \Rightarrow q_0 = \frac{Q^2}{2Mx_{Bj}}$$

$$t_\gamma \downarrow \sim \frac{1}{q_0} = \frac{2Mq^0}{Q^2 \uparrow} \qquad t_{\text{parton}} \sim \frac{1}{\Lambda}$$

(lifetime of parton)

$$t_\gamma < t_{\text{parton}}$$

free onshell particles



$$(\xi P+q)^2 = -Q^2 + 2\xi P\cdot q = 0$$

$$\Rightarrow \xi = \frac{Q^2}{2P\cdot q}$$

two center of mass energy:

(5) $$S = (k+P)^2 = 2k\cdot P + m_e^2 + \overset{\text{the mass of proton}}{M^2} \approx 2k\cdot P$$

(When energy is very high, we can neglect M_e and M)

$$\hat{S} = (q+P)^2 = -Q^2 + 2P\cdot q$$

$$\Rightarrow Q^2 + \hat{S} = 2P\cdot q = 2P\cdot k\cdot y = Sy$$

$$\because x_{Bj} = \frac{Q^2}{2P\cdot q} \qquad \therefore Q^2 = x_{Bj}\, yS$$

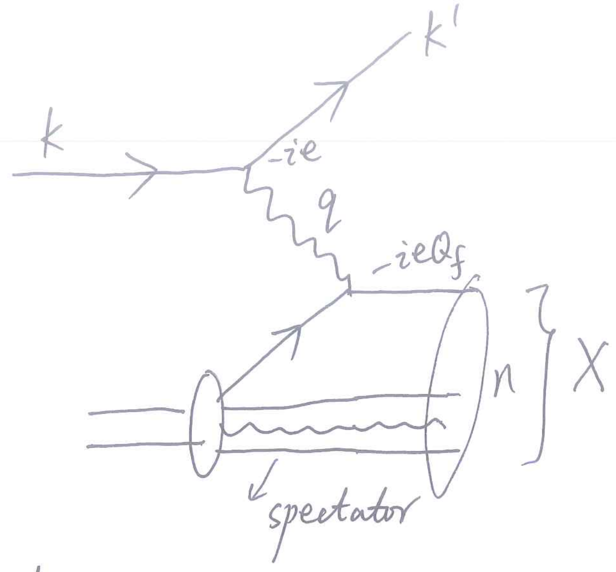
DIS cross section:

from Peskin. chapter 4

$$d\sigma_X = \frac{1}{|V_A - V_B|}\cdot\frac{1}{2E_A}\cdot\frac{1}{2E_B}\sum^{n}\int\prod_{i=1}^{n}\frac{d^3P_i}{2P_i(2\pi)^3}\cdot\frac{1}{4}|M|^2\frac{d^3k'}{(2\pi)^3 2E'} \cdot (2\pi)^4\delta^{(4)}(p+q-P_n)$$

(have used massless limit)

P_n → sum of momentum of all final particles.



$$\frac{d\sigma_X}{d^3k'} = ?$$

$$iM = \frac{(-ie)^2(-i)g^{\mu\nu}}{q^2}\,\bar{u}(k')\gamma_\mu u(k)\langle X|J_\nu(0)|P\rangle$$

(not consider Q_f for the moment)

$$= \frac{+ie^2}{q^2}\,\bar{u}(k')\gamma_\mu u(k)\underbrace{\langle X|J^\mu(0)|P\rangle}_{Q_f\,\bar{u}(\xi p+q)\gamma^\mu u(\xi p)}$$

non-perturbative part

Do computation in target rest frame:

$\frac{d\sigma_X}{d^3k'}$ is not lorentz invariance, change it to lorentz invariance.

$$d^3k' = E'^2dE'd\Omega = 2\pi E'^2dE'd\cos\theta$$

In the rest frame of proton, one can easily show

$$Q^2 = 4EE'\sin^2\frac{\theta}{2} = 2EE'(1-\cos\theta) \Rightarrow Q^2(\theta, E') \Rightarrow dQ^2(\theta, E')$$

$$dQ^2 = 2EE'd\cos\theta$$

$$\because x \equiv \frac{Q^2}{2P\cdot q} \Rightarrow x(\theta, E')$$

$$\therefore dE'd\cos\theta \xrightarrow{\text{change}} \underbrace{dx\,dQ^2}_{\text{Lorentz invariance}}$$

Working in the target rest frame:

$$V_A - V_B = 1 - 0 = 1$$

$$2E_A = 2M_P \rightarrow \text{proton mass}$$

$$2E_B = 2E$$

$$d\sigma_X = \frac{1}{2M_p}\cdot\frac{1}{2E}\sum_n \int \prod_{i=1}^{n} \frac{d^3P_i}{2P_i(2\pi)^3}\cdot\frac{1}{4}|M|^2\frac{d^3k'}{(2\pi)^3 2E'}(2\pi)^4\delta^{(4)}(p+q-P_X)$$

the leptonic tensor:

$$L_{\mu\nu} = \frac{1}{2}Tr[\not{k}'\gamma_\mu \not{k}\gamma_\nu] = 2[k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}k\cdot k']$$

the hadronic tensor:

$$W^{\mu\nu} \equiv \frac{1}{4M}\cdot\frac{1}{2\pi}\sum_s\sum_X\int\prod_i^n\frac{d^3P_i}{(2\pi)^3 2P_i}(2\pi)^4\delta^{(4)}(P+q-P_X)$$

$$\cdot\langle P,S|J^\mu(0)|X\rangle\langle X|J^\nu(0)|P,S\rangle$$

∵ any operator $\phi(x)$ in Heisenberg picture,

$$\phi(x) = \phi(\vec{x},t) = e^{i\hat{H}t}\phi(\vec{x})e^{-i\hat{H}t}$$

$$\downarrow$$

$$J^\mu(x) = e^{i\hat{P}\cdot x}J^\mu(0)e^{-i\hat{P}\cdot x}$$

$$\therefore \int d^4x\, e^{iqx}\langle P,s|J^\mu(x)|X\rangle$$

$$= \int d^4x\, e^{iqx}\langle P,s|e^{i\hat{P}\cdot x}J^\mu(0)e^{-i\hat{P}\cdot x}|X\rangle$$

$$= \int d^4x\, e^{iqx}e^{iP\cdot x}e^{-iP_X\cdot x}\langle P,s|J^\mu(0)|X\rangle$$

$$= \int d^4x\, e^{i(p+q-P_X)\cdot x}\langle P,s|J^\mu(0)|X\rangle$$

$$= (2\pi)^4\delta^{(4)}(p+q-P_X)\langle P,s|J^\mu(0)|X\rangle$$

$$\int\prod_i^n\frac{d^3P_i}{(2\pi)^3 2P_i}$$

$$\therefore W^{\mu\nu} = \frac{1}{4M}\cdot\frac{1}{2\pi}\int d^4x\, e^{iq\cdot x}\sum_s\sum_X\langle P,s|J^\mu(x)|X\rangle\langle X|J^\nu(0)|P,s\rangle$$

$$= \frac{1}{4\pi M}\int d^4x\, e^{iq\cdot x}\frac{1}{2}\sum_s\langle P,s|J^\mu(x)J^\nu(0)|P,s\rangle$$

where

$$\sum_{X(n)}\int\prod_{i=1}^n\frac{d^3P_i}{(2\pi)^3 2P_i}|X\rangle_n\,{}_n\langle X| = 1$$

$$\equiv \frac{1}{4\pi M}\int d^4x\, e^{iq\cdot x}\langle P|J^\mu(x)J^\nu(0)|P\rangle$$

where last line defines an abbreviated notation for the spin-averaged proton state. After integrating over (X), one obtains

$$\Rightarrow \frac{d\sigma_X}{d^3k'} = \frac{\alpha_{EM}^2}{EE'Q^4} L_{\mu\nu} W^{\mu\nu} \qquad \alpha_{EM} = \frac{e^2}{4\pi}$$

From QED

equal { current conservation $q_\mu W^{\mu\nu}=0$ $q_\nu W^{\mu\nu}=0$
ward identity
gauge invariance

without loss of generality,

$$W^{\mu\nu} = -W_1\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)+\frac{W_2}{\underset{\text{proton mass}}{M_p^2}}\left(P^\mu-\frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu-\frac{P\cdot q}{q^2}q^\nu\right)$$

W_1. W_2 ⟶ two scalar functions, not tensor.

$$\frac{d\sigma_X}{d^3k'} = \frac{2\alpha^2}{EE'Q^4}\left[2k\cdot k' W_1 + \frac{2P\cdot k\,P\cdot k'}{M^2}W_2\right]$$

where use $\frac{2k\cdot q\,k'\cdot q}{q^2} = k\cdot k'$ & $q=k-k'$

try to do in Lorentz invariance way

$$\left|\frac{\partial(x,\overset{Q^2}{\overset{\uparrow}{y}})}{\partial(E',\cos\theta)}\right| = \frac{2E'}{ys} \qquad x_{Bj}\,y\,Q^2 = s$$

by change variables

$$\rightarrow \frac{d\sigma}{dx\,dQ^2}$$

W_1. W_2 have dimension, $\dim[W^{\mu\nu}] = -1$, the dimension of inverse mass.

redefine: $\underset{\text{dimensionless}}{F_1} \equiv \underset{\text{mass}}{M} W_1 \qquad \underset{\text{dimensionless}}{F_2} \equiv \dfrac{Q^2}{2\underset{\text{mass}}{M} x_{Bj}} W_2$

W_1, W_2 → depend on frame.

$$\Rightarrow \frac{d\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[y^2 F_1 + \frac{(1-y)}{x_{Bj}} F_2\right]$$ ☆

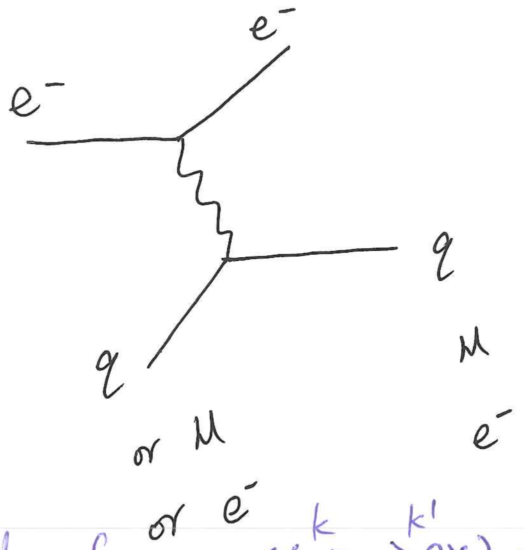
note: → later ignore Bj, refer to x_{Bj}, most of time x is x_{Bj}.

→ parton density (parton distribution)

$f_f(\xi)$: Probability of finding constituent f with logitudinal fraction ξ.

f → u, d, s …

$\xi = x_{Bj}$ (at lowest order)



charge is different. (Q_f)

therefore, $\sigma(\overset{k}{e}p \to \overset{k'}{e}X) = \int d\xi\, f_f(\xi)\,\sigma(ef \to ef)$.

$$\frac{d\sigma}{d\hat{t}} = \frac{2\pi\alpha^2 Q_f^2}{\hat{s}^2}\left[\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}\right] \qquad \hat{s} + \hat{u} + \hat{t} = 0$$

try to write down

$$\frac{d\sigma}{dx\,dQ^2}$$

two steps:

$$\sigma = \sum_i \int d\overset{\nearrow x_{Bj}}{\xi}\, \underset{\substack{\downarrow \\ \text{all possible quark}}}{f_i}(\xi)\, d\hat{t}\, \overset{\nearrow (t\ \text{channel})}{\frac{d\sigma}{d\hat{t}}}(e^- q \to e^- q)$$

$$\frac{d\sigma}{dx\,dQ^2} = \sum_i Q_f^2 \frac{\overset{2}{\cancel{4}}\pi\alpha^2}{Q^4} \underline{[1+(1-y)^2]} f(\xi)\cdot\frac{1}{\cancel{2}}$$

(in parton model)

In most general way,

$$\frac{d\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[y^2 F_1 + \frac{(1-y)}{x_{Bj}} F_2\right]$$

use $y^2 = [1-(1-y)]^2 = 1+(1-y)^2 - 2(1-y)$

$$\Rightarrow \frac{d\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q} F_1 \underline{[1+(1-y)^2]} + (1-y)\left[\frac{F_2}{x} - 2F_1\right]$$

compare

$$\Rightarrow \frac{F_2}{x} - 2F_1 = 0$$

Callan-Gross relation

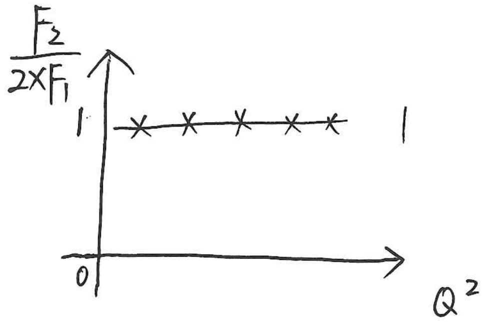
Conclusion: ① $F_2 = 2xF_1$ (for parton model)

② $F_1 = \sum_i Q_i^2 \cdot \frac{1}{2} f_i(\xi)$

$F_2 = 2xF_1 = \sum_i Q_i^2\, x f(x)$

* C-G relation depends on two theory $\begin{cases} \text{parton model} \\ \text{treating quark as femion} \end{cases}$

F_1, F_2 can only depend on x or $\frac{M^2}{Q^2}$



very nice exp. (parton model)

if quark spin is $\frac{1}{2}$, femion, then $\frac{F_2}{2xF_1} = 1$; this figure means quark spin is $\frac{1}{2}$.

If Q^2 is large enough, $2xF_1/F_2 = 1$ spin $= \frac{1}{2}$.

$F_1, F_2 \to$ dimensionless quantity.

see $F_{1,2}\left(x, \frac{M^2}{Q^2}\right)$

if chose $Q^2 \uparrow$, $\frac{M^2}{Q^2} \doteq 0 \Rightarrow F_{1,2}$ is only function of x_{Bj}.

$\boxed{\text{Bjorken scaling}}$

Bjorken scaling in high energy include quantum evolution,

$$F_{1,2}\left(x, \frac{M^2}{Q^2}, \frac{\mu^2}{Q^2}\right)$$

μ: renormalization scale

$\downarrow$ DGLAP

$\boxed{\text{violation of Bjorken scaling}}$

* explain why we have $F_2 = 2xF_1$

$\because F_2 \propto \#(\sigma_T + \sigma_L)$ (# is same)

$2xF_1 \propto \# \sigma_T$

$\therefore F_2 - 2xF_1 \propto \sigma_L$

virtual particle (virtual photon) has 3 polarization.

2 → transverse polarization, 1 longitudinal polarization

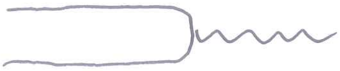
ε_T ε_L

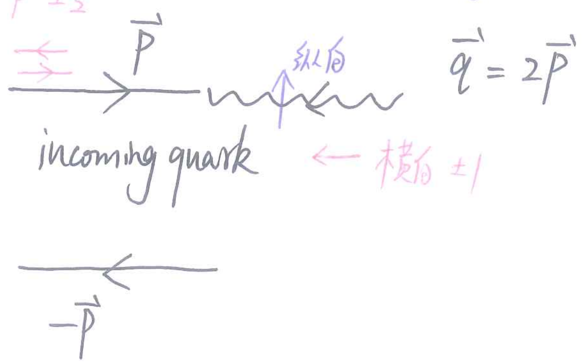
$$\underbrace{\varepsilon^T_\mu W^{\mu\nu} \varepsilon^T_\nu}_{\substack{\text{transverse } \sigma \\ (\sigma_T)}} \qquad \underbrace{\varepsilon^L_\mu W^{\mu\nu} \varepsilon^L_\nu}_{\substack{\text{longitudinal } \sigma \\ (\sigma_L)}}$$

$$F_2 - 2xF_1 \propto \sigma_L \overset{\text{always}}{=\!=} 0$$

Why? In Breit frame, (physical meaning clear)

photon like wall   Choose helicity states as eigenstates.





① quark can not absorb a longitudinally polarized photon.

② quark can absorb a head-on photon with helicity ±1.

$h_\gamma = 0$, $\sigma_T \neq 0$

$\sigma_L = 0$

$\sigma_L = 0$ violate spin / helicity conservation.

$\sigma_L = 0$ $\overset{\frown}{\text{only}}$ hold in naive parton model.

note: ① Above calculation is boost invariant.

Bjorken frame, $(p^+, \frac{m^2}{p^+}, 0)$ $q^\mu = (q_0, q_\perp, 0)$

partonic picture is manifest.

② Dipole frame. useful in high energy scattering.

$S_\perp =$  $= S(r_\perp)$ gluon dominent

⟶ scattering amplitude