May 24, 2016

2016. 5. 24 derivation of DGLAP eq. ⑫

$F_2 - 2xF_1 \propto \sigma_L$

quark can not absorb a longitudinally polarized photon.

$\therefore\ \sigma_L = 0$

sometimes call $\sigma_T \to$ transverse obsorptive cross section

$\sigma_L \to$ longitudinal obsorptive cross section.

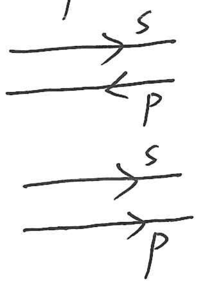
define $\sigma_\lambda = \dfrac{4\pi^2\alpha}{|q_{lab}|} \varepsilon_\mu^\lambda \varepsilon_\nu^{*\lambda} W^{\mu\nu}$

$|q_{lab}| \to$ lab system

λ stands for two kinds of polarizations.

$$\varepsilon_\mu^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm 1 \\ 0 \end{pmatrix} \qquad \varepsilon_\mu^L = \begin{pmatrix} \sqrt{\nu^2+Q^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

transverse polarization



$$q^\mu = (\nu, 0, 0, \underbrace{-\sqrt{\nu^2+Q^2}}_{-z \text{ direction}}) \qquad |q_{lab}| = \left|\sqrt{\nu^2+Q^2}\right|$$

incoming state polarization must be the same. namely, eg $\varepsilon_\mu^\lambda \cdot \varepsilon_\nu^{*\lambda}$

at last,
$$\begin{cases} 2xF_1 = \dfrac{Q^2}{4\pi^2\alpha}\sigma_T \\ F_2 = \dfrac{Q^2}{4\pi\alpha}(\sigma_T + \sigma_L) \end{cases}$$

$\sigma_T = \frac{1}{2}[\sigma(+)+\sigma(-)]$ $\quad \sigma_L = \sigma_\lambda = 0$

average cross section

$\because F_1 \propto \sigma_T \,,\; F_2 \propto (\sigma_T + \sigma_L)$

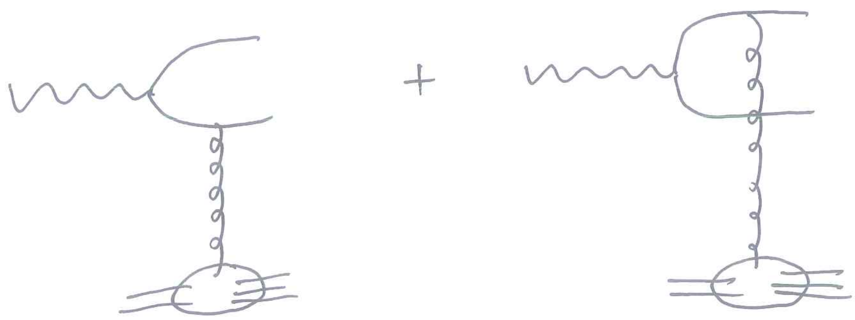
$$\frac{d\sigma_{DIS}}{dx\,dQ^2} \xlongequal{\text{generally}} \frac{\alpha}{\pi Q^2}\left[y^2\frac{\sigma_T}{2x} + \frac{(1-y)}{x}(\sigma_T+\sigma_L)\right]$$

note: σ_T, σ_L are very useful in small-x physics. In small-x, always talk about σ_T, σ_L.

In naive parton model, $\sigma_L = 0$, only true in naive parton model.

In general, it is not true ($\because$ have gluon).

eg. $\gamma^* g \to q\bar{q}$, $\sigma_L \neq 0$

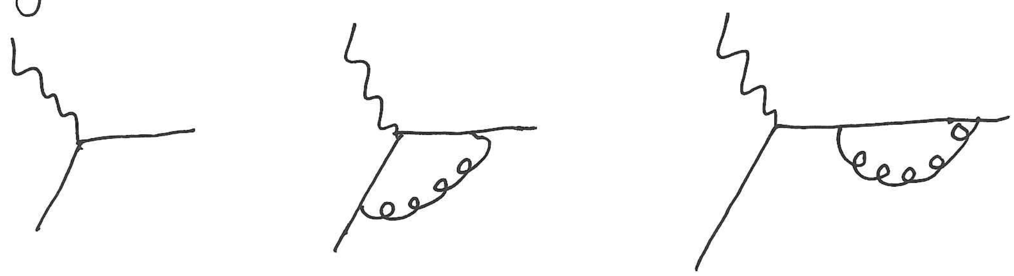


in general $\sigma_L \ll \sigma_T$

sometimes define $F_L = F_2 - 2xF_1 \longrightarrow$ longitudinal structure function.

* Derivation of DGLAP eq.

1. Why need DGLAP eq. ?



(depend on gauge choice)

gluon can not interact with photon directly.
Don't distinguish gluon from q or $\bar{q}$.

$\Rightarrow$ DGLAP eq. can renormalize quark and gluon distribution.
DIS σ_{tot} is not divergent.

2. How does F_2 look like?

at one-loop order:

DGLAP + corrections.

$$F_2 = \underbrace{\sum_i Q_i^2\, x f_i(x, Q^2)}_{\text{(LO result)}} + \underbrace{\frac{\alpha_s}{2\pi} \sum_i Q_i^2\, x \int_x^1 \frac{dz}{z} \left[q\left(\frac{x}{z}, Q^2\right) \tilde{f}^{q}(z) + g\left(\frac{x}{z}, Q^2\right) \tilde{f}^{g}(z) \right]}_{\text{(correction)}}$$

$f_i(x, Q^2)$: parton distribution function.

$\tilde{f}(z)$: coefficient
$\downarrow$ finite

DGLAP + redefined parton distribution function $\Rightarrow$ F_2 finite

q_0 : bare quark distribution function
define tree level
use naive parton model

dimensional regularization: $4 \to 4-2\varepsilon$

$$\begin{cases} MS \\ \overline{MS} \end{cases}$$

$\overline{MS}$:

$$q(x,Q^2) = q_0(x) - \frac{\alpha}{2\pi} \cdot \frac{1}{\varepsilon} \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$\cdot \int dy\, dz\, \delta(x-yz) \left[P_{qq}(z) q(y) + P_{qg}\, g(y) \right]$$

technic: $\frac{1}{\varepsilon} a^{\varepsilon} = \frac{1}{\varepsilon} + \ln a + \varepsilon + \cdots$

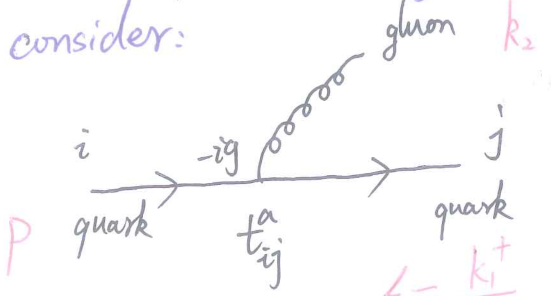
3. DGLAP eq.

reference: Altralli & Parisi NPB 1977 (298)

Here the derivation is slightly different with this reference.
Here we use light-cone coordinates and light-cone gauge (physical gauge).

In LCPT: (use light-cone perturbation theory)

consider:



$\xi = \frac{k_1^+}{p^+}$

vertex: → gluon polarization vector

$$-ig\, \bar{u}(k_1) \gamma \cdot \varepsilon\, u(p)$$

average & sum over color:

$$\frac{1}{N_c} tr[t^a t^a] = C_F$$

momentum in light-cone coordinate:

$$p^\mu = (\overset{+}{p^+}, \overset{-}{0}, \overset{\perp}{0_\perp})$$

$$k_1^\mu = \left(\xi p^+, \frac{k_\perp^2}{\xi p^+}, +k_\perp\right)$$

$$k_2^\mu = \left((1-\xi) p^+, \frac{k_\perp^2}{(1-\xi) p^+}, -k_\perp\right)$$

square above amplitude: (not include color for the moment)

$$\frac{1}{2}\sum_{spin}(-ig)\bar{u}(k_1)\gamma\cdot\varepsilon\, u(p)\cdot(ig)\bar{u}(p)\gamma\cdot\varepsilon^* u(k_1)$$

$$=\frac{1}{2}\sum_{spin}(-ig)\bar{u}(k_1)\gamma^\mu\cdot\varepsilon_\mu u(p)(ig)\bar{u}(p)\gamma^\nu\varepsilon^*_\nu u(k_1)$$

$$=\frac{1}{2}g^2\,Tr[\not{k}_1\gamma^\mu\not{p}\gamma^\nu]\sum_\lambda\varepsilon_\mu(\lambda)\varepsilon^*_\nu(\lambda)$$

$$=\frac{1}{2}g^2\,Tr[k_{1\rho}\gamma^\rho\gamma^\mu p_\sigma\gamma^\sigma\gamma^\nu]\sum_\lambda\varepsilon_\mu(\lambda)\varepsilon^*_\nu(\lambda)$$

$$=\frac{1}{2}g^2 k_{1\rho}p_\sigma\,Tr[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu]\sum_\lambda\varepsilon_\mu\varepsilon^*_\nu$$

$$=\frac{1}{2}\times 4g^2 k_{1\rho}p_\sigma(g^{\rho\mu}g^{\sigma\nu}-g^{\rho\sigma}g^{\mu\nu}+g^{\mu\sigma}g^{\nu\rho})\sum_\lambda\varepsilon_\mu\varepsilon^*_\nu$$

$$=2g^2[k_1^\mu p^\nu-k_1\cdot p\,g^{\mu\nu}+k_1^\nu p^\mu]\sum_\lambda\varepsilon_\mu\varepsilon^*_\nu \qquad (\star 1)$$

gluon polarization: $\varepsilon_\mu=\left(\varepsilon^+=0,\ \frac{-2k_\perp\varepsilon_\perp}{(1-z)p^+},\ \varepsilon_\perp^\pm\right)$ note: in light-cone gauge

$$k_2^\mu\cdot\varepsilon_\mu=0$$

Have two ways to do calculation $(\star 1)$

note: $\sum_\lambda\varepsilon_\lambda^{\mu\perp}\varepsilon_\lambda^{*\perp}=-g_\perp^{\mu\nu}$, $\sum\varepsilon^i_{\lambda\perp}\varepsilon^i_{\lambda\perp}=\delta^{ij}$

(1) put ε_μ into expression $(\star 1)$.

(2) use: in LCPT $\sum_\lambda\varepsilon^\mu(\lambda)\varepsilon^{*\nu}(\lambda)=-g^{\mu\nu}+\frac{\eta^\mu k_2^\nu+\eta^\nu k_2^\mu}{k_2^+}$

in LC gauge $\eta=(\overset{+}{0},\ \overset{-}{2},\ \overset{\perp}{0_\perp})$

Here we use second method:

namely: $2g^2[k_1^\mu p^\nu-k_1\cdot p\,g^{\mu\nu}+k_1^\nu p^\mu]\sum_\lambda\varepsilon_\mu\varepsilon^*_\nu$

$$=2g^2[k_1^\mu p^\nu-k_1\cdot p\,g^{\mu\nu}+k_1^\nu p^\mu]\left[-g_{\mu\nu}+\frac{\eta_\mu k_{2\nu}+\eta_\nu k_{2\mu}}{k_2^+}\right]$$

$$=2g^2\Big[-k_1\cdot p+4k_1\cdot p-k_1\cdot p+\frac{(k_1\cdot\eta)(p\cdot k_2)}{k_2^+}-\frac{(k_1\cdot p)(\eta\cdot k_2)}{k_2^+}+\frac{(k_1\cdot k_2)(p\cdot\eta)}{k_2^+}$$
$$+\frac{(p\cdot\eta)(k_1\cdot k_2)}{k_2^+}-\frac{(k_1\cdot p)(\eta\cdot k_2)}{k_2^+}+\frac{(k_1\cdot\eta)(p\cdot k_2)}{k_2^+}\Big]$$

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$$= 2g^2\left[2\frac{\overset{\xi p^+}{(k_1\cdot\eta)}\overset{\frac{1}{2}\frac{k_\perp^2}{(1-\xi)}}{(p\cdot k_2)}}{k_2^+} + 2\cdot\frac{\overset{p^+}{(p\cdot\eta)}(k_1\cdot k_2)}{k_2^+} - 2\frac{\overset{\frac{k_\perp^2}{2\xi}}{(k_1\cdot p)}\overset{(1-\xi)p^+}{(\eta\cdot k_2)}}{k_2^+} + 2k_1\cdot p\right]$$

$\because\ p\cdot\eta = 2p^+\cdot\frac{1}{2} = p^+$

$k_1\cdot\eta = \xi p^+$

$k_1\cdot p = \frac{1}{2}p^+\cdot\frac{k_\perp^2}{\xi p^+} = \frac{k_\perp^2}{2\xi}$

$\eta\cdot k_2 = (1-\xi)p^+$

$p\cdot k_2 = \frac{1}{2}\frac{k_\perp^2}{1-\xi}$

$$k_1\cdot k_2 = \frac{1}{2}\left(\xi p^+\cdot\frac{k_\perp^2}{(1-\xi)p^+}\right) + \frac{1}{2}\cdot\frac{k_\perp^2}{\xi p^+}(1-\xi)p^+ + k_\perp^2$$

$$= \frac{1}{2}\frac{\xi}{1-\xi}k_\perp^2 + \frac{1}{2}\frac{(1-\xi)}{\xi}k_\perp^2 + k_\perp^2$$

$k_2^+ = (1-\xi)p^+$

$$= 2g^2\left[2\cdot\frac{\xi p^+\cdot\frac{1}{2}\frac{k_\perp^2}{(1-\xi)}}{(1-\xi)p^+} + 2\cdot\frac{p^+\left(\frac{1}{2}\cdot\frac{\xi}{1-\xi}k_\perp^2 + \frac{1}{2}\frac{(1-\xi)}{\xi}k_\perp^2 + k_\perp^2\right)}{(1-\xi)p^+}\right.$$

$$\left. -2\,\frac{\frac{k_\perp^2}{2\xi}\cdot(1-\xi)p^+}{(1-\xi)p^+} + 2\cdot\frac{k_\perp^2}{2\xi}\right]$$

$$= 2g^2\left[\frac{\xi k_\perp^2}{(1-\xi)^2} + \frac{1}{1-\xi}\left(\frac{\xi}{1-\xi} + \frac{1-\xi}{\xi} + 2\right)k_\perp^2 - \frac{k_\perp^2}{\xi} + \frac{k_\perp^2}{\xi}\right]$$

$$= 2g^2\left[\frac{\xi}{(1-\xi)^2} + \frac{\xi}{(1-\xi)^2} + \frac{1}{\xi} + \frac{2}{1-\xi}\right]k_\perp^2$$

$$= 2g^2\,\frac{\xi^2 + \xi^2 + (1-\xi)^2 + 2(1-\xi)\xi}{(1-\xi)^2\xi}k_\perp^2$$

$$= 2g^2\,\frac{\cancel{2\xi^2} + 1 + \xi^2 \cancel{-2\xi} \cancel{+2\xi} \cancel{-2\xi^2}}{(1-\xi)^2\xi}k_\perp^2$$

$$= 2g^2\cdot\frac{1+\xi^2}{\xi(1-\xi)^2}k_\perp^2 = \boxed{g^2\cdot\frac{2k_\perp^2}{\xi(1-\xi)}\cdot\frac{1+\xi^2}{1-\xi}}$$

In LCPT, light-cone propagator (light-cone denominator)

$$\frac{1}{\sum_{in} p^- - \sum_{out} p^-} = \frac{1}{0 - \frac{k_\perp^2}{\xi p^+} - \frac{k_\perp^2}{(1-\xi)p^+}} = \frac{-\xi(1-\xi)p^+}{k_\perp^2}$$

$$\therefore \left|\frac{1}{\sum_{in} p^- - \sum_{out} p^-}\right|^2 = \frac{\xi^2(1-\xi)^2 p^{+2}}{k_\perp^4} = \left|\frac{1}{\frac{k_\perp^2}{\xi(1-\xi)p^+}}\right|^2$$

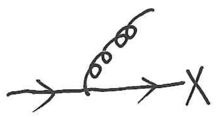
full contribution:

$$\left|\frac{1}{\sqrt{p^+k_1^+k_2^+}}\right|^2 \cdot g^2 C_F \cdot \frac{2k_\perp^2}{\xi(1-\xi)} \frac{1+\xi^2}{1-\xi} \times \frac{\xi^2(1-\xi)^2 p^{+2}}{k_\perp^4}$$

prefactor

$$= \frac{1}{p^+} g^2 C_F \cdot 2 \cdot \frac{1+\xi^2}{1-\xi} \cdot \frac{1}{k_\perp^2}$$

phase space: if we measure quark, we need integral gluon momentum.



$$\int \frac{dk_2^+ d^2k_\perp}{16\pi^3}$$

then, we have

$$\int \frac{dk_2^+ d^2k_\perp}{8\cancel{16}\pi^3} \cdot \frac{1}{p^+} g^2 \cdot C_F \cancel{2} \frac{1+\xi^2}{1-\xi} \cdot \frac{1}{k_\perp^2}$$

$$= \int d\xi\, d^2k_\perp \frac{\alpha_s C_F}{2\pi^2} \cdot \left(\frac{1+\xi^2}{1-\xi}\right)_+ \cdot \frac{1}{k_\perp^2}$$

$\rightsquigarrow$ add virtual graph

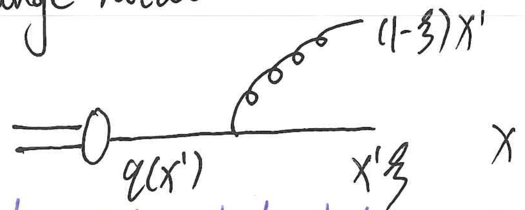
change variable:

$$\int \frac{dk_2^+}{p^+} = \int d\xi$$

$$\frac{g^2}{8\pi^2} = \frac{\alpha_s}{2\pi}$$

$$\left(\alpha_s = \frac{g^2}{4\pi}\right)$$

change notation a little



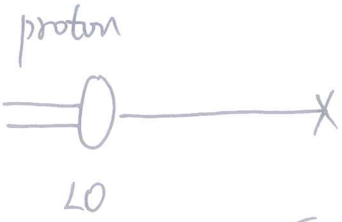
$$x'\xi = x$$

$$x' = \frac{x}{\xi} \Rightarrow \text{always } x' > x$$

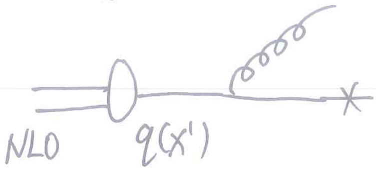
$\xi \to$ splitting fraction

Consider total quark distribution carried by $q(x)$ $[x, x+dx]$

$$q(x) = q_0(x) + \int dx' \int d\xi \int_{\mu^2}^{Q^2} d^2k_\perp \frac{\alpha_s C_F}{2\pi^2} \frac{1+\xi^2}{1-\xi} \frac{1}{k_\perp^2} \cdot q(x')\, \delta(x'\xi - x)$$



$q(x) = q_0(x)$



$x'\xi = x$

$$q(x) = q_0(x) + \int dx' \int d\xi \int d^2k_\perp \frac{\alpha_s C_F}{2\pi^2} \frac{1+\xi^2}{1-\xi} \frac{1}{k_\perp^2} q(x') \delta(x'\xi - x)$$

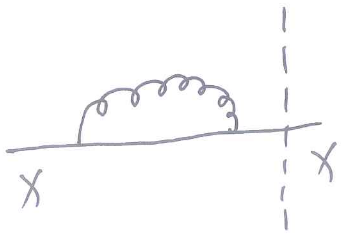
(real diagram)

For real diagram, $x'\xi = x \Rightarrow x' = \frac{x}{\xi} \leq 1 \Rightarrow \xi \geq x$

$$\therefore \int_x^1 d\xi \left(\frac{1+\xi^2}{1-\xi}\right) q(x')$$

$f(x) = \delta(1-x) + \frac{\alpha}{2\pi} C_F \ln\frac{Q^2}{M^2}\left(\frac{1+\xi^2}{1-\xi}\right)_+$ $\quad \int dx\, f(x) = 1$ ☆

Virtual correction:



note: Splitting function is the same with real diagram.
phase space is the same with real diagram.
But virtual propagator has a -1, and
$q(x') \longrightarrow q(x)$

real contribution: $\int_x^1 d\xi \left(\frac{1+\xi^2}{1-\xi}\right) \cdot q(x')$ depend on ξ

virtual contribution: $-\int_0^1 d\xi \left(\frac{1+\xi^2}{1-\xi}\right) q(x)$

$\boxed{\text{plus function}}$: $\int_x^1 d\xi\, [f(\xi)]_+ g(\xi) \equiv \int_x^1 d\xi\, f(\xi) g(\xi) - \int_0^1 d\xi\, f(\xi) g(1)$

then, $\int_x^1 d\xi \left(\frac{1+\xi^2}{1-\xi}\right) q(x') - \int_0^1 d\xi \left(\frac{1+\xi^2}{1-\xi}\right) q(x) = \int_x^1 d\xi \left(\frac{1+\xi^2}{1-\xi}\right)_+ q(x')$

real: $\frac{\alpha_s C_F}{2\pi^2}\int_x^1 \frac{d\xi}{\xi}\,\frac{(1+\xi^2)}{1-\xi}\, q\left(\frac{x}{\xi}\right)\int\frac{d^2k_\perp}{k_\perp^2}$

virtual: $-\frac{\alpha_s C_F}{2\pi^2}\int_0^1 d\xi\left(\frac{1+\xi^2}{1-\xi}\right)\cdot q(x)\int\frac{d^2k_\perp}{k_\perp^2}$

real + virtual:

no divergence

have divergence

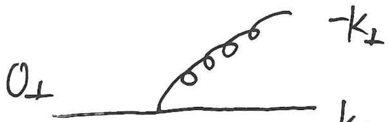
$$\frac{\alpha_s C_F}{2\pi^2}\underbrace{\int_x^1 d\xi\left(\frac{1+\xi^2}{1-\xi}\right)_+}_{\triangleq P_{qq}}\frac{1}{\xi}\,q\left(\frac{x}{\xi}\right)\int_{\Lambda^2}^{Q^2}\frac{d^2k_\perp}{k_\perp^2}$$

quark splitting function

$\Lambda^2\to 0$, have divergence (infra div.)

$Q^2\to\infty$, UV divergence

(by UV conterterm cancel)



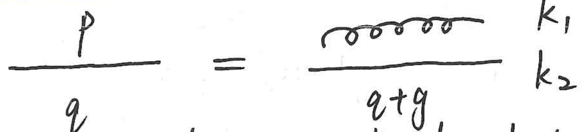
$k_\perp\to 0$ collinear

$\int_{\Lambda^2}^{Q^2}\frac{d^2k_\perp}{k_\perp^2}$ have collinear divergence

collinear divergence is understandable. The theory is called $\boxed{\text{KLN theory}}$.

idea: IR divergence appears because some of the states are "degenerate", but we treat them differently.

in QFT,



$p = k_1 + k_2$

When we introduce quark distribution function, we have broken this "degenerate". This is the origin of this divergent.

correct this problem:

assume $\int_{\Lambda^2}^{Q^2}\frac{d^2k_\perp}{k_\perp^2} = \pi\ln\frac{Q^2}{\Lambda^2} = \pi\ln\frac{Q^2}{M_f^2} + \pi\ln\frac{M_f^2}{\Lambda^2}$

absorbed it into bare quark distribution $q_0(x)$.

$\Lambda\to 0$, have divergence.

$q_{tot}(x)$ can be finite.

We can choose M_f arbitrarily. For convenience, often chose

$M_f^2 = Q^2$.

if $q(x, Q^2)$, that is to say, change q to depend on Q^2.

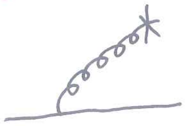
then $\frac{\partial q(x,Q^2)}{\partial \ln Q^2}$

From above, we also know quark splitting function P_{qq}

$$P_{qq}(\xi) = C_F\left(\frac{1+\xi^2}{1-\xi}\right)_+ = C_F\left[\frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi)\right]$$

where use $\left(\frac{1+\xi^2}{1-\xi}\right)_+ = \frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi)$

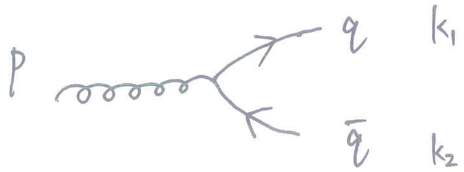
$\boxed{\text{Comment}}$ on other splitting functions:



$$P_{gq}(\xi) = C_F\frac{1+(1-\xi)^2}{\xi}$$

(note: don't need to calculate, just change $\xi \to 1-\xi$ for P_{qq})

no divergence, $\because$ can detect gluon, then the momentum of gluon must be finite.



$\left.\begin{matrix} g \to q \\ g \to \bar{q} \end{matrix}\right\}$ equal

$\frac{1}{2}tr[\not{k}_1\gamma^\mu \not{k}_2 \gamma^\nu]\sum_\lambda \varepsilon_\mu^{(\lambda)}\varepsilon_\nu^{*(\lambda)}$

$= \frac{1}{2}(2k_\perp^2)\frac{\xi^2+(1-\xi)^2}{\xi(1-\xi)} \Rightarrow P_{qg} = \frac{1}{2}[\xi^2+(1-\xi)^2]$

P_{qg} : the method is almost the same with P_{qq}. Technic is the same. vertex is different.

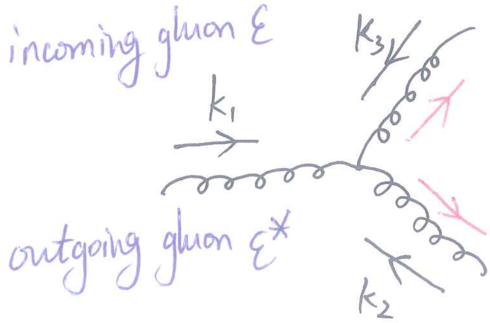
$$P_{qg} = P_{\bar{q}g} = \underbrace{\frac{1}{2}}_{\text{color factor}}[\xi^2+(1-\xi)^2]$$

P_{qg} & $P_{\bar{q}g}$ are similar to P_{qq} calculation!

no divergence.

P_{gg} : non-trivial, three-gluon vertex.

$\varepsilon_\perp = \left(\frac{1}{\sqrt{2}}, \frac{\pm i}{\sqrt{2}}\right)$



$$= -igf_{abc}[\overset{k_1+k_2}{(k_1-k_2)}\cdot\varepsilon_3^*\,\varepsilon_1\cdot\varepsilon_2^* + \overset{-k_2+k_3}{(k_2-k_3)}\cdot\varepsilon_1\,\varepsilon_2^*\cdot\varepsilon_3^* + \overset{-k_3-k_1}{(k_3-k_1)}\cdot\varepsilon_2^*\,\varepsilon_1\cdot\varepsilon_3^*]$$

color factor: $\frac{1}{N_c^2-1}\underbrace{f_{abc}f_{abc'}}_{N_c\delta_{cc'}}\delta_{cc'} = \frac{N_c(N_c^2-1)}{N_c^2-1} = N_c$

don't consider the coefficient $-igf_{abc}$ for the moment,

define $V \equiv [(k_1+k_2)\cdot\varepsilon_3^*](\varepsilon_1\cdot\varepsilon_2^*) + [(-k_2+k_3)\cdot\varepsilon_1](\varepsilon_2^*\cdot\varepsilon_3^*)$
$\quad + [(-k_3-k_1)\cdot\varepsilon_2^*](\varepsilon_1\cdot\varepsilon_3^*)$

$\because k_1 = (p^+, 0, 0) \qquad \varepsilon_1 = (0, 0, \varepsilon_\perp^\pm)$

$k_2 = (\xi p^+, \frac{k_\perp^2}{\xi p^+}, k_\perp) \qquad \varepsilon_2 = (0, \frac{2k_\perp\cdot\varepsilon_\perp^\pm}{\xi p^+}, \varepsilon_\perp^\pm)$

$k_3 = ((1-\xi)p^+, \frac{k_\perp^2}{(1-\xi)p^+}, -k_\perp) \qquad \varepsilon_3 = (0, \frac{-2k_\perp\cdot\varepsilon_\perp^\pm}{(1-\xi)p^+}, \varepsilon_\perp^\pm)$

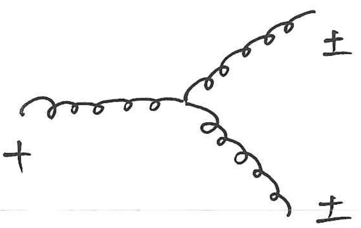
$\varepsilon_\perp^\pm = (\frac{1}{\sqrt{2}}, \frac{\pm i}{\sqrt{2}})$

consider $\frac{1}{2}\sum_{spins}|V|^2$

average incoming gluon spin

do a simple way: incoming gluon positive polarization contribution is equal to its negative polarization contribution. So we only need to consider incoming gluon positive polarization, at the same time the result multiplies by 2. then $\frac{1}{2}\times 2 = 1$

average incoming gluon spin



$V_{G+\to G\pm G\pm}$

$$|V|^2 = |V_{+++}|^2 + |V_{++-}|^2 + |V_{+-+}|^2 + |V_{+--}|^2$$

$|V_{+--}|^2 \Rightarrow 0$

$|V_{+++}|^2 \Rightarrow \left|2k_\perp\cdot\varepsilon_\perp^+\left(\frac{1}{\xi}+\frac{1}{1-\xi}\right)\right|^2$

$|V_{+-+}|^2 \to \left|2k_\perp\cdot\varepsilon_\perp^-\left(\frac{1}{1-\xi}-1\right)\right|^2$

$|V_{++-}|^2 \to \left|2k_\perp\cdot\varepsilon_\perp^-\left(\frac{1}{\xi}-1\right)\right|^2$

eg. $V_{+++} = [(k_1+k_2)\cdot\varepsilon_3^*](\varepsilon_1\cdot\varepsilon_2^*)$

$\quad + [(-k_2+k_3)\cdot\varepsilon_1](\varepsilon_2^*\cdot\varepsilon_3^*)$

$\quad + [(-k_3-k_1)\cdot\varepsilon_2^*](\varepsilon_1\cdot\varepsilon_3^*)$

$\left[-\frac{1+\xi}{1-\xi}k_\perp\varepsilon_\perp^*(+) - k_\perp\cdot\varepsilon_\perp^*(+)\right] \cdot \left(-\varepsilon_\perp(+)\varepsilon_\perp^*(+)\right)$, $\varepsilon_\perp(+)\varepsilon_\perp^*(+) = 1$

$$= \left[\left((1+\xi)p^+, \frac{k_\perp^2}{\xi p^+}, k_\perp\right)\cdot\left(0, \frac{-2k_\perp\cdot\varepsilon_\perp^*(+)}{(1-\xi)p^+}, \varepsilon_\perp^*(+)\right)\right]\left[\left(0,0,\varepsilon_\perp(+)\right)\left(0, \frac{2k_\perp\cdot\varepsilon_\perp^*(+)}{\xi p^+}, \varepsilon_\perp^*(+)\right)\right]$$

$\frac{1}{2}$

$$+ \left[\left(-\xi p^+ + (1-\xi)p^+, -\frac{k_\perp^2}{\xi p^+} + \frac{k_\perp^2}{(1-\xi)p^+}, -2k_\perp\right)\cdot\left(0,0,\varepsilon_\perp(+)\right)\right]$$

$$\cdot\left[\left(0, \frac{2k_\perp\cdot\varepsilon_\perp^*(+)}{\xi p^+}, \varepsilon_\perp^*(+)\right)\cdot\left(0, \frac{-2k_\perp\varepsilon_\perp^*(+)}{(1-\xi)p^+}, \varepsilon_\perp^*(+)\right)\right]$$

$-\varepsilon_\perp^*(+)\varepsilon_\perp^*(+) = 0$

$$+ \left\{\left[-(1-\xi)p^+ - p^+, -\frac{k_\perp^2}{(1-\xi)p^+}, +k_\perp\right]\cdot\left(0, \frac{2k_\perp\cdot\varepsilon_\perp^*(+)}{\xi p^+}, \varepsilon_\perp^*(+)\right)\right\}$$

$$\cdot\left[\left(0,0,\varepsilon_\perp(+)\right)\left(0, \frac{-2k_\perp\varepsilon_\perp^*(+)}{(1-\xi)p^+}, \varepsilon_\perp^*(+)\right)\right]$$

$$\left(\frac{1}{\cancel{2}}(\xi-2)\cancel{p^+}\cdot\frac{\cancel{2}k_\perp\cdot\varepsilon_\perp^*(+)}{\xi\cancel{p^+}} - k_\perp\varepsilon_\perp^*(+)\right)\cdot\big(-\underbrace{\varepsilon_\perp(+)\varepsilon_\perp^*(+)}_{1}\big) \Leftrightarrow \frac{2}{\xi}k_\perp\varepsilon_\perp^*(+)$$

$$= 2k_\perp\cdot\varepsilon_\perp^*(+)\left(\frac{1}{1-\xi}+\frac{1}{\xi}\right)$$

$$\therefore |V_{+++}|^2 = \left|2k_\perp\cdot\varepsilon_\perp^*(+)\left(\frac{1}{1-\xi}+\frac{1}{\xi}\right)\right|^2 = 4\left(k_\perp\cdot\varepsilon_\perp^*(+)\right)^2\left(\frac{1}{1-\xi}+\frac{1}{\xi}\right)^2$$

$$= 4\left(\frac{1}{\sqrt{2}}k_\perp^1 + \frac{i}{\sqrt{2}}k_\perp^2\right)^2\left(\frac{1}{1-\xi}+\frac{1}{\xi}\right)^2$$

$$= 4\cdot\frac{1}{2}k_\perp^2\left(\frac{1}{1-\xi}+\frac{1}{\xi}\right)^2$$

$$= 2k_\perp^2\left(\frac{1}{1-\xi}+\frac{1}{\xi}\right)^2$$

others is similar.

$|V_{++-}|^2 = 2k_\perp^2\left(\frac{1}{\xi}-1\right)^2 \qquad |V_{+-+}|^2 = 2k_\perp^2\left(\frac{1}{1-\xi}-1\right)^2 \qquad |V_{+--}|^2 = 0$

$$\therefore |V|^2 = \frac{4k_\perp^2}{\xi^2(1-\xi)^2}\left[1-\xi(1-\xi)\right]^2$$

will be cancle by phase space integration at last.

Sometimes write $\frac{2k_\perp^2}{\xi(1-\xi)}\cdot 2\left[\frac{1-\xi}{\xi}+\frac{\xi}{1-\xi}+\xi(1-\xi)\right] \xrightarrow[N_c]{\text{add color factor}} P_{gg}^{real}$

play a trick:

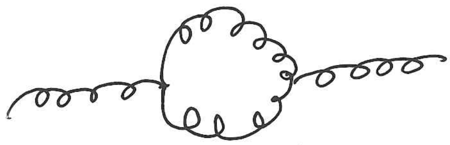
$$\frac{2\xi(1-\xi)+(1-\xi)^2+\xi^2+\xi^2(1-\xi)^2-2\xi(1-\xi)}{\xi(1-\xi)} = \frac{[1-\xi(1-\xi)]^2}{\xi(1-\xi)}$$

above part is the real contribution for P_{gg}.

Now consider virtual part for P_{gg}.

two ways $\begin{cases} \text{brute force} \\ \text{momentum conservation} \end{cases}$

quark loop



gluon loop

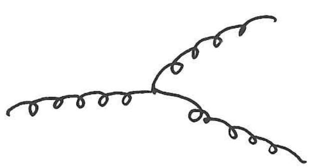
we have known $P_{gg}^{real} = 2N_c\left[\frac{1-\xi}{\xi} + \frac{\xi}{1-\xi} + \xi(1-\xi)\right]$

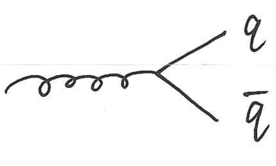
use momentum conservation:

$$\underbrace{\int_0^1 d\xi\, \xi\, \delta(1-\xi)}_{=1} + \frac{\alpha_s}{2\pi}\int_0^1 d\xi\left[P_{gg}(\xi)\xi + P_{qg}(\xi)\xi\right] = 1$$

$$\Rightarrow \int_0^1 d\xi\, \xi\left[P_{qg}(\xi) + P_{gg}(\xi)\right] = 0$$

$\delta(1-\xi)$ LO contribution





$$P_{gg}^{total} = 2N_c\left[\frac{1-\xi}{\xi} + \frac{\xi}{(1-\xi)_+} + \xi(1-\xi)\right] + \underbrace{\delta(1-\xi)N_V}_{\text{virtual}}$$

Similarly

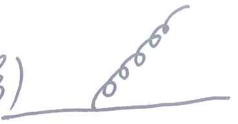
For $P_{qq}(\xi)$,

by the way, can check $\left(\frac{1+\xi^2}{1-\xi}\right)_+ = \frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi)$

quark number:

For virtual contribution $\xi = 1$

For real contribution $0 \leq \xi \leq 1$

$\delta(1-\xi)$  1

quark pdf

No matter how many gluons radiated, quark number is 1 before and after gluon radiated.

$$\underbrace{\int_0^1 d\xi\, \delta(1-\xi)}_{\substack{LO \\ \parallel \\ 1}} + \underbrace{\frac{\alpha_s C_F}{2\pi} \int_0^1 d\xi \left(\frac{1+\xi^2}{1-\xi}\right)_+}_{\substack{\parallel \\ 0}} = 1$$

From plus function: $\int_x^1 d\xi\, (f(\xi))_+ g(\xi) = \int_x^1 d\xi\, f(\xi) g(\xi) - \int_0^1 d\xi\, f(\xi)\, g(1)$

$\Rightarrow$ if $g(\xi) = \text{const}$, $g(\xi) = g(1)$, we will get $\int_0^1 d\xi \left(\frac{1+\xi^2}{1-\xi}\right)_+ = 0$

In addition, according to quark number conservation $\int_0^1 d\xi\, P_{qq}(\xi) = 0$

$\Rightarrow$ the coefficient of $\delta(1-\xi)$ of P_{qq} is $\frac{3}{2}$.

$$\int_0^1 d\xi\, P_{qq}(\xi) = 0 \Rightarrow \int_0^1 \Big(\underbrace{\frac{1+\xi^2}{1-\xi}}_{\text{real}} + \underbrace{\#}_{\text{virtual}}\Big) d\xi = 0$$

or $\int_0^1 \left(\frac{1+\xi^2}{1-\xi} - \frac{2}{1-\xi} + \frac{2}{1-\xi} + \#\right) d\xi = 0$

or $\int_0^1 \frac{\xi^2 - 1}{1-\xi} d\xi + \int_0^1 \frac{2}{1-\xi} d\xi + \int_0^1 \# \, d\xi = 0$

or $-\int_0^1 (\xi + 1)\, d\xi + \int_0^1 \frac{2}{1-\xi} d\xi + \int_0^1 \# \, d\xi = 0$

or $-\frac{3}{2} + \int_0^1 \frac{2}{1-\xi} d\xi + \int_0^1 \# \, d\xi = 0$

$\Rightarrow \int_0^1 \# \, d\xi = \frac{3}{2} - \int_0^1 \frac{2}{1-\xi} d\xi$ substitute into

$\Rightarrow \int_0^1 \frac{1+\xi^2}{1-\xi} d\xi - \int_0^1 \frac{2}{1-\xi} d\xi + \frac{3}{2} = 0$

or $\int_0^1 \frac{1+\xi^2}{(1-\xi)_+} d\xi + \frac{3}{2} = 0$

$\Downarrow$

$$P_{qq}(z) = C_F\left[\frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi)\right]$$