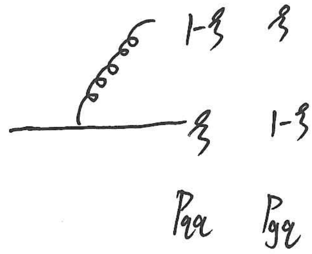
May 26, 2016. DGLAP

$\boxed{\text{the splitting functions}}$:

$$P_{qq} = C_F\left(\frac{1+\xi^2}{1-\xi}\right)_+ = C_F\left[\frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi)\right]$$



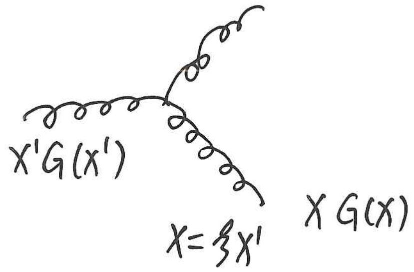
$$P_{gq} = C_F\,\frac{1+(1-\xi)^2}{\xi} \qquad \text{no virtual part}$$

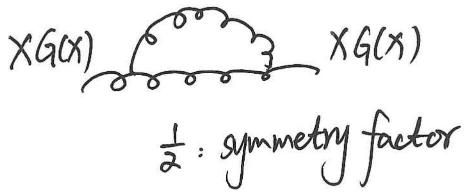
$$P_{qg} = P_{\bar{q}g} = \frac{1}{2}\left[\xi^2+(1-\xi)^2\right]$$

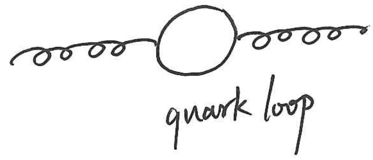
$$P_{gg} = 2N_c\left[\frac{\xi}{(1-\xi)_+} + \frac{1-\xi}{\xi} + \xi(1-\xi)\right] + 2N_c\cdot\frac{11-2n_f/N_c}{12}\,\delta(1-\xi)$$

$N_c \to C_A$

$N_c = C_A$: number of color







compute the coefficient of $\delta(1-\xi)$ of P_{gg} :

have two ways $\begin{cases} \text{① brute force} \\ \text{② momentum conservation} \end{cases}$

brute force:

one virtual: $$-N_c\int_0^1 \frac{1}{2}\,d\xi\cdot 2\left[\frac{\xi}{1-\xi} + \frac{1-\xi}{\xi} + \xi(1-\xi)\right] X\,G(X) \qquad (1)$$

real : $$N_c\int_X^1 d\xi\; 2\left[\frac{\xi}{1-\xi} + \frac{1-\xi}{\xi} + \xi(1-\xi)\right] X'\,G(X') \qquad (2)$$

where $\int_X^1 d\xi\,\delta(1-\xi)\,X'G(X') = XG(X)$

another virtual: $$\int_0^1 d\xi\cdot\frac{n_f}{2}\left[\xi^2+(1-\xi)^2\right] = \frac{n_f}{2}\cdot\frac{2}{3} = \frac{n_f}{3}. \qquad (3)$$

For $\frac{1}{2}\int_0^1 d\xi\left[\frac{\xi}{1-\xi} + \frac{1-\xi}{\xi} + \xi(1-\xi)\right]$

$$= \frac{1}{2}\int_0^1 d\xi\left[\frac{1}{1-\xi} - 1 + \frac{1}{\xi} - 1 + \xi(1-\xi)\right]$$

this two integral is actually same.

$$= \frac{1}{2}\int_0^1 d\xi \left[\frac{2}{1-\xi} - 1 + (-1) + \xi(1-\xi)\right]$$

$$= \frac{1}{2}\int_0^1 d\xi \left[\frac{2}{1-\xi} - 2 + \xi - \xi^2\right]$$

$$= \frac{1}{2}\int_0^1 \frac{2}{1-\xi} d\xi - \frac{11}{12}$$

$$= \int_0^1 \frac{1}{1-\xi} d\xi - \frac{11}{12}$$

at last, using (1), (2), (3) and plus function definition $\Rightarrow$

the coefficient of $\delta(1-\xi)$ of P_{gg}.

momentum conservation method:

LO: real contribution $\xrightarrow{\text{straightforward}}$ virtual contribution

① $\int_0^1 d\xi\, P_{qq} = 0$ (quark number conservation)

real — virtual

gain particle — loss particle

Note: negative contribution can be viewed as annihilation probability! (loss term)

$$\int d\xi \left[\delta(\xi-1) + \underbrace{\frac{\alpha_s}{2\pi} P(\xi) \ln Q^2/\Lambda^2}_{\text{one loop correction}}\right] = 1 \Rightarrow \int d\xi\, P_{qq} = 0$$

② $\int_0^1 d\xi\, P_{gg} \neq 0$ (gluon number is not conserved, but longitudinal momentum is conserved.)

③
$$\begin{cases} \int_0^1 d\xi \left[\xi P_{qq} + \xi P_{gq}\right] = 0 \\ \int_0^1 d\xi \left[\xi P_{qg} \cdot \underset{q \text{ or } \bar{q}}{\underset{\downarrow}{2}} n_f + \xi P_{gg}\right] = 0 \end{cases}$$

Using $\int_0^1 d\xi\,[\xi P_{qg}\cdot 2n_f + P_{gg}\cdot \xi] = 0$

explicitly

$$n_f\cdot \cancel{\tfrac{1}{2}}\times\cancel{2}\int_0^1 d\xi\;\xi[\xi^2+(1-\xi)^2] + 2N_c\int_0^1 d\xi\cdot\xi\left[\frac{\xi}{(1-\xi)_+}+\frac{1-\xi}{\xi}+\xi(1-\xi)\right] + N_V = 0$$

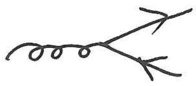
where $n_f\int_0^1 d\xi\cdot\xi\,[\xi^2+(1-\xi)^2] = \frac{n_f}{3}$

$$\int_0^1 d\xi\cdot\xi\left[\frac{\xi}{(1-\xi)_+}+\frac{1-\xi}{\xi}+\xi(1-\xi)\right] = -\frac{11}{12}$$

$$\therefore\; N_V - \frac{11}{12}\cdot 2N_c + \frac{n_f}{3} = 0 \;\Rightarrow\; N_V = 2N_c\left[\frac{11}{12} - \frac{2n_f/N_c}{12}\right]$$

The DGLAP evolution equations:

define the flavor nonsinglet distribution function



$$\Delta = \sum_f (q_f - q_{\bar f})$$ no contribution from gluon

flavor singlet distribution: $\Sigma = \sum_f [q_f + q_{\bar f}]$

DGLAP renormalized eq. or physical eq.

$$\overset{\Delta}{\uparrow}\hspace{-0.5em}xq = x\overset{\Delta}{\uparrow}\hspace{-0.5em}q_0 + \frac{\alpha_s}{2\pi}\int_x^1\frac{d\xi}{\xi}P_{qq}\underbrace{\int_{\Lambda^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}}_{=\ln\frac{Q^2}{\Lambda^2}}\cdot\frac{x}{\xi}\,\overset{\Delta}{\uparrow}\hspace{-0.5em}q\left(\frac{x}{\xi}\right) \qquad (4)$$

$$+\frac{\alpha_s}{2\pi}\cdots P_{qg}\ln\frac{Q^2}{\Lambda^2}\,\frac{x}{\xi}\,g\left(\frac{x}{\xi}\right)$$

$\Lambda^2\to 0$ divergent.

$$\ln\frac{Q^2}{\Lambda^2} = \ln\frac{Q^2}{M_f^2} + \underbrace{\ln\frac{M_f^2}{\Lambda^2}}_{\text{absorbed into PDF}}$$

usually choose $M_f^2 = Q^2$.

For (4), taking derivative

$$\frac{d x \Delta}{d \ln Q^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} P_{qq}(\xi) \frac{x}{\xi} \Delta\left(\frac{x}{\xi}\right)$$

naive parton model

$$\int_0^1 x^w dx \frac{d \Delta(x, Q^2)}{d \ln Q^2} = \int_0^1 dx\, x^w \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} P_{qq}(\xi) \frac{1}{\xi} \Delta\left(\frac{x}{\xi}, Q^2\right)$$

* quark and gluon coupled each other: (DGLAP evolution equations)

$$\frac{d}{d \ln Q^2} \begin{pmatrix} x\Sigma(x, Q^2) \\ xG(x, Q^2) \end{pmatrix} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \begin{pmatrix} x\Sigma(x, Q^2) \\ xG(x, Q^2) \end{pmatrix}$$

* the solution of the DGLAP equations:

$\boxed{\text{transform}}$:

Mellin transform: $f_w(Q^2) \equiv \int_0^1 dx\, x^w f(x, Q^2)$

Inverse Mellin transform: $f(x, Q^2) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f_w(Q^2)\, x^{-w-1}\, dw$

$$\Delta(x, Q) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} P_{qq} \ln\frac{Q^2}{\Lambda^2} \Delta\left(\frac{x}{\xi}\right)$$

$$\frac{d \Delta_w}{d \ln Q^2} = \frac{\alpha_s}{2\pi} \int_0^1 dx\, x^w \int_x^1 \frac{d\xi}{\xi} P_{qq}(\xi) \Delta\left(\frac{x}{\xi}\right) \qquad (\xi > x)$$

change variable

$$\int_0^1 d\xi \int_0^\xi \frac{dx}{\xi} \frac{x^w}{\xi^w} \Delta\left(\frac{x}{\xi}\right) \xi^{w+1} P_{qq}(\xi)$$

define: $z = \frac{x}{\xi}$

$$\int_0^1 dz\, z^w \Delta(z) \xrightarrow{\text{by definition}} \Delta(w)$$

$$\therefore \frac{d\Delta_\omega(Q^2)}{d\ln Q^2} = \underbrace{\frac{\alpha_s}{2\pi}\int_0^1 d\xi\, \frac{\xi^{\omega+1}}{\xi} P_{qq}(\xi)}_{\equiv\ \gamma_{qq}} \Delta_\omega(Q^2)$$

γ_{qq} : anomalous dimension

can calculate

$$\gamma_{qq}(\omega) = C_F\left[\frac{3}{2} + \frac{1}{(1+\omega)(2+\omega)} - 2\psi(\omega+2) + 2\psi(1)\right]$$

↓ digamma function.

note that $\psi(1) = -\gamma_E$, with γ_E Euler's constant.

then DGLAP eq. will change from a differential-integral eq. to a differential equation.

$$\frac{d\Delta_\omega(Q^2)}{d\ln Q^2} = \frac{\alpha_s}{2\pi}\gamma_{qq}(\omega)\,\Delta_\omega(Q^2)$$

solution: $\Delta_\omega(Q^2) = \Delta_\omega(Q_0^2)\, e^{\overbrace{\boxed{\frac{\alpha_s}{2\pi}\gamma_{qq}(\omega)}}^{D} \ln\frac{Q^2}{Q_0^2}}$ fixed coupling.

initial condition: $Q^2 = Q_0^2$.

$$= \Delta_\omega(Q_0^2)\left(\frac{Q^2}{Q_0^2}\right)^{\frac{\alpha_s}{2\pi}\gamma_{qq}(\omega)}$$

rewrite

$$= \Delta_\omega(Q_0^2)\left[1 + D\ln\frac{Q^2}{Q_0^2} + \frac{1}{2}D^2\ln^2\frac{Q^2}{Q_0^2} + \cdots\right]$$

$$\left(\frac{\alpha_s}{2\pi}\ln\frac{Q^2}{Q_0^2}\right)^n$$

* General solution needs to diagonalize the matrix.

Through the diagonalization, we can also solve DGLAP eqs.

$$S^{-1}\frac{d}{d\ln Q^2}\left(\quad\right)S = \left(\quad\right)\begin{pmatrix}\Sigma\\ G\end{pmatrix}$$

↓ diagonal

Using inverse Mellin transform, $\Delta_\omega(Q^2) \longrightarrow \Delta(x)$

* What happens at small-x or large Q^2?

From DGLAP evolution equations, we can see at small-x, the ξ-integral may get extra enhancement from the small-ξ region.

$$\begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix}$$

$$P_{gg}\,,\ P_{gq} \propto \frac{1}{\xi}\Big|_{\xi\to 0}$$

At low-x limit, gluon distribution becomes important.

$$P_{qq}\,,\ P_{qg} \to \text{finite}\Big|_{\xi\to 0}$$

gluon distribution is inhanced by $\frac{1}{\xi}\Big|_{\xi\to 0}$

Using the approximation for the gluon-gluon splitting function $P_{gg}\propto \frac{1}{\xi}\Big|_{\xi\to 0}$, We can write down an evolution eq. for gluon distribution:

$$\frac{dG}{d\ln Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \underset{\downarrow\ \frac{2N_c}{\xi}}{P_{gg}} \frac{d\xi}{\xi}\, G\left(\frac{x}{\xi}\right)$$

low x, $\xi\to 0$, $P_{gg} = \frac{2N_c}{\xi}$

do many tranformations:

$$\frac{dG_w}{d\ln Q^2} = \frac{2N_c\alpha_s}{2\pi}\int_0^1 \frac{dx}{\xi}\,\frac{x^w}{\xi^w}\int\frac{d\xi}{\xi}\cdot\frac{1}{\xi}\, G\left(\frac{x}{\xi}\right)\xi^{w+1}$$

$$= \frac{\cancel{2}N_c\alpha_s}{\cancel{2}\pi}\, G_w\cdot\frac{1}{\omega} \qquad \int_0^1 d\xi\cdot\xi^{w-1} = \frac{1}{\omega}$$

$$= \frac{N_c\alpha_s}{\pi}\, G_w\cdot\frac{1}{\omega}$$

Solution of G_w: $G_w(Q^2) = G_w(Q_0^2)\, e^{\frac{\alpha_s N_c}{\pi}\ln\frac{Q^2}{Q_0^2}\cdot\frac{1}{\omega}}$

Using inverse Mellin transformation $\Longrightarrow$ gluon distribution

$$G(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\omega \, x^{-\omega-1} G_\omega(Q_0^2)\, e^{\underbrace{\frac{\alpha_s N_c}{\pi}\cdot\frac{1}{\omega}\ln\frac{Q^2}{Q_0^2}+\omega\ln\frac{1}{x}}_{\equiv h(\omega)}}$$

small-x.

Using saddle point approximation.

$$x^{-\omega} = \frac{1}{x^\omega} = e^{\omega\ln\frac{1}{x}}$$

$$h'(\omega)=0 \quad \Longrightarrow \quad \omega_{sp} = \sqrt{\frac{\alpha_s N_c \ln\frac{Q^2}{Q_0^2}}{\pi\ln\frac{1}{x}}}$$

to approximate the exponent $h(\omega)$ by its Taylor expansion around the saddle point up to the quadratic term

$$h(\omega) \doteq h(\omega_{sp}) + \frac{1}{2}h''(\omega_{sp})(\omega-\omega_{sp})^2$$

first derivative $h'(\omega)=0$.

$$\omega - \omega_{sp} \overset{\text{must be}}{=\!=} \text{imaginary number}$$

$$\omega - \omega_{sp} \equiv i\bar{\omega}$$

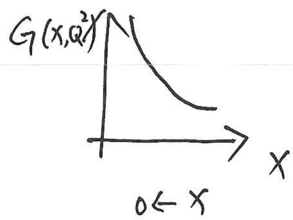
(fixed coupling constant)

$$G(x,Q^2) = \# \, e^{2\sqrt{\frac{\alpha_s N_c}{\pi}\underbrace{\ln\frac{Q^2}{Q_0^2}}_{\text{large}}\underbrace{\ln\frac{1}{x}}_{\text{large}}}}$$

resummation

DLA DGLAP solution

$x\downarrow,\ G(x,Q^2)\uparrow.$



resum $\ln\frac{1}{x} \rightarrow$ BFKL $\quad \left(\alpha_s\ln\frac{1}{x}\right)^n.$