May 30, 2016

From DGLAP to BFKL

Paradigm shift

$\downarrow$ Balitsky-Fadin-Kuraev-Lipatov

DGLAP	BFKL
$s \sim Q^2 \to \infty$	$s \gg Q^2$
① $\underbrace{\alpha_s}_{\text{small}} \underbrace{\ln \frac{Q^2}{Q_0^2}}_{\text{large}} \to$ not so large	$\alpha_s \ln \frac{1}{x}$
with fixed x (not so small)	with fixed Q^2
perturbative theory.	$x \ll 1$, $x \to 0$, $s \to \infty$
② collinear approximation $k_\perp \sim 0$ or integrated over; ladder diagram with ordered $k_\perp$	assume: t-channel gluon exchange. unintegrated $k_\perp$

$qq \to qq$

center of mass frame

$\hat{s} \to \infty$

$\hat{s} + \hat{u} + \hat{t} = 0$

$\hat{s} \sim |-\hat{u}| \gg -\hat{t}$

$\therefore \underbrace{\frac{\hat{s}^2 + \hat{u}^2}{\hat{s}^2}}_{\text{can ignore}} + \frac{1}{\hat{t}^2} \to$ dominant.

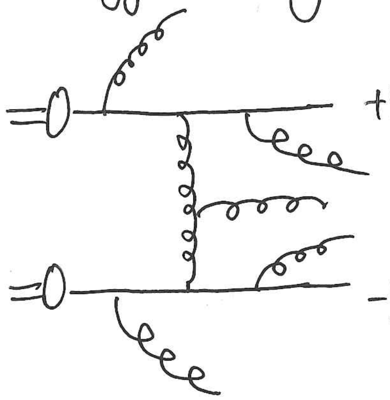
③ There are 2 different but equivalent formulation of BFKL evolution, i.e.

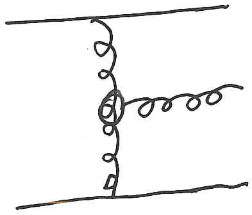
- momentum formulation $\to$ complicated
- coordinate formulation $\to$ simpler

See Y. Kovchegov's book

Comment on momentum formulation (BFKL)

High energy scattering



$\Rightarrow$  Lipatov vertex.

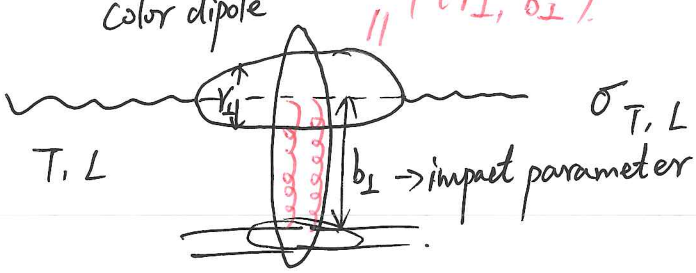
④ In the double logarithmic approximation

$$\alpha_s \ln\frac{Q^2}{Q_0^2} \ln\frac{1}{x} \gtrsim 1$$

$$\text{DGLAP} \Longleftrightarrow \text{BFKL}$$

DIS cross section, derive momentum formulation BFKL

In DIS: $F_2 = \dfrac{Q^2}{4\pi^2\alpha_{em}} \cdot (\sigma_T + \sigma_L)$



$r \gg$ size of proton

$$\sigma_L = \sum_f Q_f^2 x (2N_c) \int_0^1 dz \int \frac{d^2b_\perp d^2r_\perp}{(2\pi)^2} \cdot 4z^2(1-z)^2 \cdot K_0(\epsilon_f r_\perp) T(r_\perp, b_\perp)$$

small-x limit, have so-called dipole frame.

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2\alpha}(\sigma_T + \sigma_L)$$

$$\sigma_T = \sum_f Q_f^2 x (2N_c) \int_0^1 dz \int \frac{d^2b_\perp d^2r_\perp}{(2\pi)^2} \left\{ [z^2 + (1-z)^2] \epsilon_f^2 K(\epsilon_f r_\perp) \right\} T(r_\perp, b_\perp)$$

choose photon momentum $q = (q_0, 0, 0, q_3)$

boost, rotate from Breit frame $\rightarrow$ dipole frame

Y. Kovchegov & E. Levin book derived F_2 in coordinate space.

$$F_2 = \frac{Q^2}{4\pi\alpha_{em}} \cdot 2N_c \int_0^1 dz \int \frac{d^2b_\perp\, d^2r_\perp}{(2\pi)^2} \sum_f Q_f^2$$

$\rightarrow$ depend on $q\bar{q}$ flavor

$$\cdot \left\{ [z^2+(1-z)^2]\, \varepsilon_f^2 K_1^2(\varepsilon_f r_\perp) + 4z^2(1-z)^2 Q^2 K_0^2(\varepsilon_f r_\perp) \right\}$$

splitting function

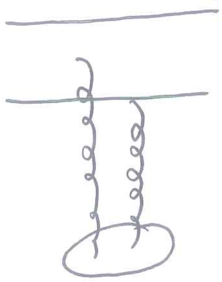
$$\cdot \underbrace{[1 - S(r_\perp, b_T)]}_{\text{sometimes write } T(r_\perp, b_\perp)}$$

$\sigma_L + \sigma_T$

$$\varepsilon_f^2 = z(1-z)Q^2$$

transform into coordinate space:

in coordinate space



depend on size of dipole: $r_\perp = 0$, interaction can not happen.

dipole how far from the target : $b_\perp$

coordinate space $\rightarrow$ momentum space : two different.

but at last, this two is the same through Fourier transform.

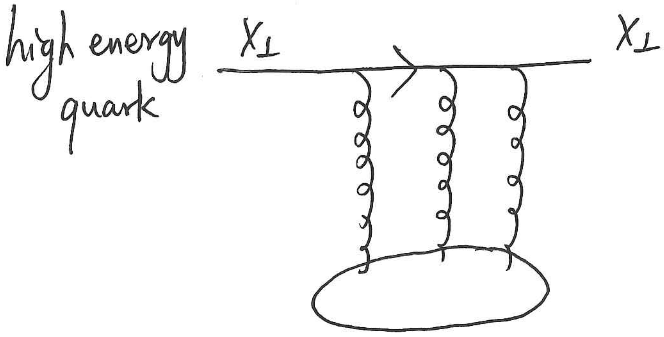
see arXiv: 1101.0715

* comment momentum space $\rightarrow$ coordinate space in dealing with BFKL.

coordinate space advantage
- ① picture is quite simple
- ② easy to do resummation in coordinate space (as long as high energy limit, have eikonal approximation)

eikonal approximation:

(It is very convenient to use coordinate space formulation).



several collisions

moving in high energy: $v_\perp = \frac{k_\perp}{p^+}$

range of interaction: L

$$\Delta x_\perp = v_\perp \cdot L$$

quantum fluctuation: $\frac{1}{k_\perp}$

$$x_\perp = \frac{1}{k_\perp}$$

at very high energy: $\frac{\Delta x_\perp}{x_\perp} = \frac{k_\perp^2 L}{p^+} \ll 1$

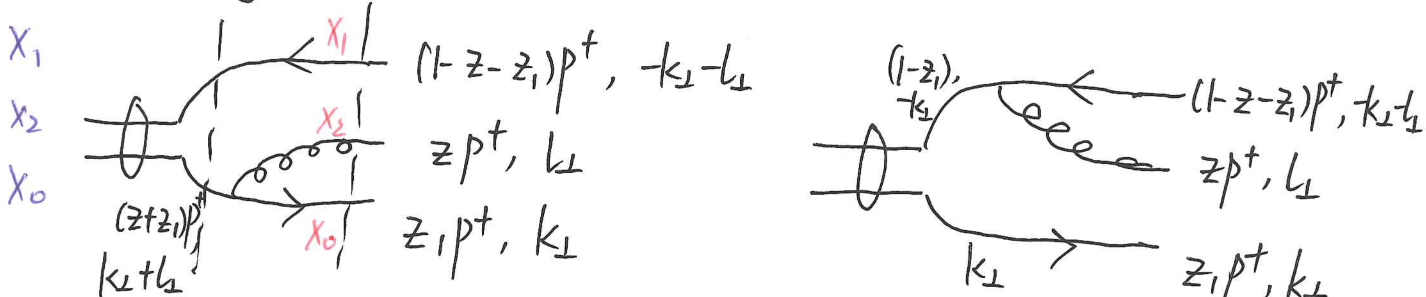
the $x_\perp$ of the high energy particle is not changed.

In DIS, $\frac{Q^2}{s} ML \Rightarrow x_{Bj} ML \ll 1$

$x_{Bj} \cdot s \ll 1$, $x_{Bj} \ll \frac{1}{s}$

* BFKL evolution:

low-x gluon



color dipole

$$z_1 \gg z, \quad 1 - z_1 \gg z$$

$x_{01} = x_0 - x_1$

$x_{02} = x_0 - x_2$

sum over these two diagrams.

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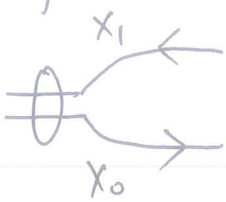
First define



$1-z_1, -k_\perp$

$z_1, k_\perp$

$\psi(k_\perp, z_1)$

define ψ

first in momentum space

normalize of this ψ: $\int_0^1 dz_1 d^2k_\perp |\psi(k_\perp, z_1)|^2 = 1$

$$\psi(x_{01}, z_1) = \int d^2k_\perp \, e^{-ik_\perp \cdot x_{01}} \psi(k_\perp, z_1)$$

can prove $\int_0^1 dz_1 \int d^2k_\perp |\psi(k_\perp, z_1)|^2 = 1 = \int_0^1 dz_1 \int \frac{d^2x_{01}}{(2\pi)^2} |\psi(x_{01}, z_1)|^2$

For the first diagram, according to LCPT:

$$\underbrace{\frac{1}{\sqrt{(z+z_1)p^+ \, zp^+ \, z_1p^+}}}_{\text{flux}} (+igt^a) \bar{u}(k_1) \gamma\cdot\varepsilon \, \underbrace{u(k_1+l)}_{\text{very small}} \left(\frac{zp^+}{l_\perp^2}\right) \psi(k_\perp + l_\perp, z+z_1)$$

$$\frac{zp^+}{l_\perp^2} \Leftarrow \frac{1}{\frac{(k_\perp+l_\perp)^2}{(z+z_1)p^+} - \frac{k_\perp^2}{z_1p_1^+} - \frac{l_\perp^2}{zp^+}}$$

$\bar{u}(p) \gamma^\mu u(p) = 2p^\mu$ (at high energy limit)

dominate by $\gamma^+ \cdot p^+$

$z < z_1$ $\quad u(k_1 + l) \longrightarrow u(k_1)$

spinor part can be simplified

$$\bar{u}(k_1)\gamma\cdot\varepsilon \, u(k_1) = 2k_1\cdot\varepsilon = 2\cdot\frac{1}{2}k_1^+ \cdot \frac{2l_\perp\cdot\varepsilon_\perp}{zp^+}$$

in light-cone gauge

$$= z_1 p^+ \cdot \frac{2l_\perp\cdot\varepsilon_\perp}{zp^+}$$

$\varepsilon = \left(0, \frac{2l_\perp\cdot\varepsilon_\perp^\pm}{zp^+}, \varepsilon_\perp^\pm\right)$

$$= 2\frac{\varepsilon_\perp\cdot l_\perp}{z}\cdot z_1$$

$\because \varepsilon^+ = 0, \ \varepsilon^- \neq 0$, fixed by polarization

$$\frac{1}{\sqrt{p^{+3} z z_1 z_1}} \cdot igt^a \cdot \frac{z_1 \cdot 2 l_\perp \cdot \varepsilon_\perp}{z} \frac{z \cdot p^+}{l_\perp^2} \; \psi(k_\perp + l_\perp, z + z_1)$$

initial $q\bar{q}$ wave function

$$\psi(k_\perp + l_\perp, z+z_1) \downarrow \psi(k_\perp + l_\perp, z_1)$$

$\because z \ll z_1$, $\therefore z$ can be ignored

$$= igt^a \frac{1}{\sqrt{p^+ z}} \cdot 2 \frac{l_\perp \cdot \varepsilon_\perp}{l_\perp^2} \cdot \psi(k_\perp + l_\perp, z_1)$$

Similarly, for the second diagram:

wave function of dipole $\psi(k_\perp, z_1)$

$$\Rightarrow igt^a \frac{1}{\sqrt{p^+ z}} \frac{2 l_\perp \cdot \varepsilon_\perp}{l_\perp^2} \psi(k_\perp, z_1)$$

$$\frac{\varepsilon_\perp \cdot l_\perp}{l_\perp^2 + z(1-z)Q^2} \atop \varepsilon_f^2$$

do Fourier transform $\Longrightarrow \varepsilon_\perp \nabla_{r_\perp} K_0(\varepsilon_f r_\perp)$

sum over this two diagrams:

$$2 igt^a \frac{1}{\sqrt{p^+ z}} \frac{l_\perp \cdot \varepsilon_\perp}{l_\perp^2} \left[\psi(k_\perp + l_\perp, z_1) - \psi(k_\perp, z_1) \right]$$

gluon from $\bar{q}$.

do Fourier transform into coordinate space,

keep $2igt^a \frac{1}{\sqrt{z p^+}}$, do Fourier transform

for $\frac{l_\perp \cdot \varepsilon_\perp}{l_\perp^2} [\psi(k_\perp + l_\perp, z_1) - \psi(k_\perp, z_1)]$

$$\Rightarrow \int d^2 l_\perp d^2 k_\perp \, e^{-ik_\perp \cdot x_{01}} e^{-il_\perp x_{12}} \frac{l_\perp \cdot \varepsilon_\perp}{l_\perp^2} [\psi(k_\perp + l_\perp, z_1) - \psi(k_\perp, z_1)]$$

from momentum space $\to$ coordinate space

$$\boxed{\int d^2 l_\perp \, e^{-i l_\perp \cdot b_\perp} \frac{\varepsilon_\perp \cdot l_\perp}{l_\perp^2} = -2\pi i \frac{\varepsilon_\perp \cdot b_\perp}{b_\perp^2}}$$

useful!

$$\frac{1}{l_\perp^2 + \lambda^2}$$

prove: take partial derivative with respect to $b_\perp$, at last take limit $\lambda \to 0$.

then $\int d^2l_\perp d^2k_\perp e^{-ik_\perp\cdot x_{01}} e^{-il_\perp\cdot x_{12}} \frac{l_\perp \varepsilon_\perp}{l_\perp^2}[\psi(k_\perp+l_\perp, z_1)-\psi(k_\perp,z_1)]$

$$= \psi(x_{01}, z_1)(-2\pi i)\left[\frac{\varepsilon_\perp\cdot x_{12}}{x_{12}^2} - \frac{\varepsilon_\perp\cdot x_{02}}{x_{02}^2}\right]$$

in coordinate space

square this

$$|\psi(x_{01}, z_1)|^2 (2\pi)^2 \underbrace{\sum_\pm}_{\text{sum over polarization}} \left[\frac{\varepsilon_\perp\cdot x_{12}}{x_{12}^2} - \frac{\varepsilon_\perp\cdot x_{02}}{x_{02}^2}\right]\left[\frac{\varepsilon_\perp^* x_{12}}{x_{12}^2} - \frac{\varepsilon_\perp^* x_{02}}{x_{02}^2}\right]$$

using

$$\sum_\pm \varepsilon_\perp^* \varepsilon_\perp = -g_\perp = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$|\psi^{(1)}|^2 \equiv |\psi^{(0)}(x_{01}, z_1)|^2 (2\pi)^2 \left[\frac{x_{12}}{x_{12}^2} - \frac{x_{02}}{x_{02}^2}\right]\left[-\frac{x_{12}}{x_{12}^2} - \frac{x_{02}}{x_{02}^2}\right]$$

$$= |\psi|^2 (2\pi)^2 \left(\frac{1}{x_{12}^2} + \frac{1}{x_{02}^2} - \frac{2x_{12}\cdot x_{02}}{x_{12}^2\cdot x_{02}^2}\right)$$

$$= |\psi|^2 (2\pi)^2 \frac{(x_{12}-x_{02})^2}{x_{12}^2 x_{02}^2}$$

gluon phase space integral: $\int \frac{dl^+ dl_\perp^2}{4\pi(2\pi)^2} \longrightarrow \int \frac{dl^+ d^2x_2}{16\pi^3}$

color factor squared $\Rightarrow C_F$

after add other factors, we get

$$(4\pi)^2 g^2 C_F |\psi^{(0)}(x_{01}, z_1)|^2 \frac{(x_{12}-x_{02})^2}{x_{12}^2 x_{02}^2}$$

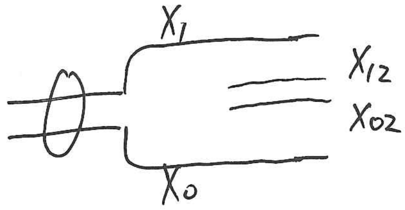
add gluon phase space integral & In large N_c limit $C_F = \frac{N_c}{2}$

$$\Rightarrow |\psi^{(1)}|^2 = \frac{\alpha_s N_c}{2\pi^2}\int \frac{dz}{z} d^2x_2 \frac{x_{01}^2}{x_{02}^2 x_{12}^2} |\psi^{(0)}|^2$$ (the correction after one gluon radiated)

$$\frac{|\psi^{(1)}|^2}{|\psi^{(0)}|^2} = \frac{\alpha_s N_c}{2\pi^2}\int \frac{dz}{z} d^2x_2 \frac{x_{01}^2}{x_{02}^2 x_{12}^2}$$

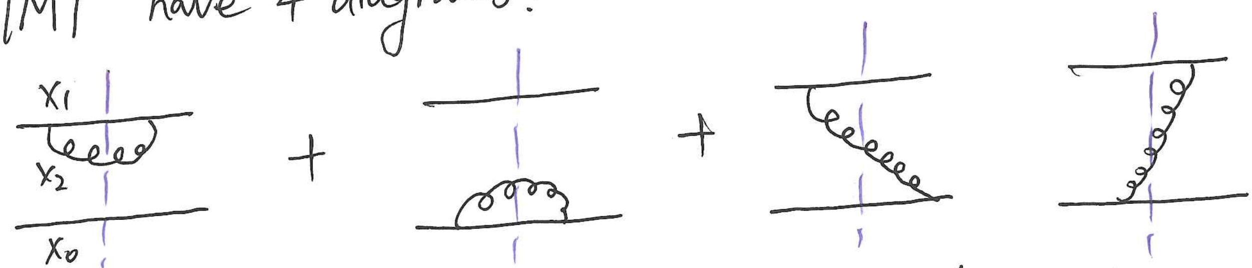
physical meaning:

the probability of one dipole splitting into two dipole.



(probability of generating two color dipoles from a parent dipole.)

$|M|^2$ have 4 diagrams:



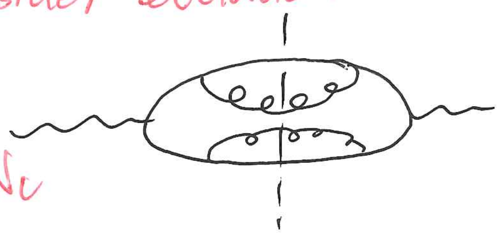
In large N_c limit, radiate one gluon $\Longleftrightarrow$ add a dipole

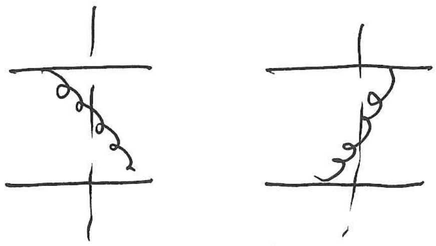
$$g \leftrightarrow q\bar{q}$$

From $$\frac{|\psi^{(1)}|^2}{|\psi^{(0)}|^2} = \frac{\alpha_s N_c}{2\pi^2}\int \frac{dz}{z}\, d^2x_2\, \frac{x_{01}^2}{x_{02}^2 x_{12}^2},$$

easily get BFKL.

Consider evolution:

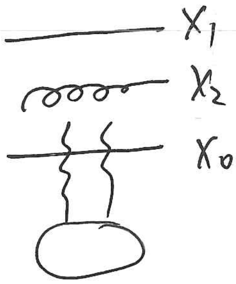




large N_c limit!

$S = 1 - T$

$=$



$$\propto \frac{\alpha_s N_c}{2\pi^2} \underbrace{\boxed{\int_x^1 \frac{dz}{z}}}_{\ln\frac{1}{x}} \int \frac{d^2x_2\, x_{01}^2}{x_{02}^2 x_{12}^2}$$

$\sigma \propto T(r_\perp, b_\perp)$

↳ forget it for the moment.

$$T = T_0 + \frac{\alpha_s}{2\pi^2} N_c \int_0^1 \frac{dz}{z} \int \frac{d^2x_2\, x_{01}^2}{x_{12}^2 x_{02}^2} \left[T(x_{12}) + T(x_{02}) - T(x_{01}) \right]$$

x → light-cone divergence

virtual diagrams.

$$\int_x^1 \frac{dz}{z} = \ln\frac{1}{x} \equiv y$$

rapidity

$$T(y) = T_0 + \frac{\alpha_s N_c}{2\pi^2}\, y \int d^2x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \left[T(x_{02}, y) + T(x_{12}, y) - T(x_{01}, y) \right]$$

original, no energy dependence.

energy dependence.

$$\frac{\partial T}{\partial y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \left[T(x_{02}, y) + T(x_{12}, y) - T(x_{01}, y) \right]$$

(BFKL)

in coordinate space.

$x_{01} = x_0 - x_1$

$x_{12} = x_1 - x_2$

$x_{02} = x_0 - x_2$

resum all loops

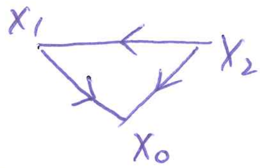
$$\sum_n \left(\alpha_s \ln\frac{1}{x}\right)^n$$

$$\alpha_s \ln\frac{1}{x}$$

↓

one loop

* Solution of BFKL equation:



From DIS, $F_2 \to T$, back to T.

Solution of BFKL equation.

$\lambda = 1+2i\nu$

technic is the same. (coordinate space or momentum space)

use the identity

$$I = \int d^2x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2}\left[\underbrace{x_{12}^\lambda + x_{02}^\lambda}_{\text{symmetric, just times 2}} - x_{01}^\lambda\right] = 4\pi x_{01}^\lambda \chi(\lambda)$$

x_{12} is the magnitude of $\vec{x}_{12}$.

where $\chi(\lambda) = \psi(1) - \frac{1}{2}\psi\left(\frac{\lambda}{2}\right) - \frac{1}{2}\psi\left(1-\frac{\lambda}{2}\right)$

for any complex number λ.

eigen function of BFKL kernel x_{12}^λ, x_{02}.

digamma function: $\psi(x) = \frac{d}{dx}\ln(\Gamma(x))$

similar to the method of solving DGLAP eq., having several technics:

① the first two terms are symmetric

$$\int d^2x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2}\cdot 2x_{12}^\lambda = 2\int_0^\infty du \int_0^{2\pi} d\theta \frac{u^{\lambda-1}}{1-2u\cos\theta+u^2}\, x_{01}^\lambda$$

$$u = \frac{|x_{12}|}{|x_{01}|}$$

last term can be written as

$$\int d^2x_2 \frac{x_{01}(x_{02}-x_{12})}{x_{12}^2 x_{02}^2}\, x_{01}^\lambda$$

that is to say: $$\int \frac{d^2x_2\, x_{01}^2}{x_{02}^2\, x_{12}^2}\cdot x_{12}^\lambda = \int_0^\infty du \int_0^{2\pi} d\theta \frac{u^{\lambda}-1}{1+u^2-2u\cos\theta}\, x_{01}^\lambda \qquad u = \frac{|x_{12}|}{|x_{01}|}$$

$$\int \frac{d^2x_2\, x_{01}^2}{x_{02}^2\, x_{12}^2}\cdot x_{02}^\lambda = \int_0^\infty du \int_0^{2\pi} d\theta \frac{u^{\lambda-1}}{1+u^2-2u\cos\theta}\, x_{02}^\lambda \qquad u = \frac{|x_{12}|}{|x_{01}|}$$

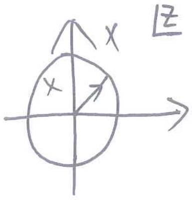
② $$\int \frac{d^2x_2\, x_{01}\overset{[x_{02}-x_{12}]}{[x_{01}]}}{x_{12}^2\, x_{02}^2} = \int_0^{\infty} du \int d\theta \frac{2\cos\theta}{1+u^2-2u\cos\theta}$$

③ For any integral

$$\int_0^{2\pi} d\theta \frac{1}{1+\varepsilon\cos\theta} = \frac{2\pi}{\sqrt{1-\varepsilon^2}}\Big|_{\varepsilon<1}$$

contour integral

$$z=e^{i\theta},\quad dz=iz\,d\theta,\quad d\theta=\frac{dz}{iz}$$



two poles, find residue

$$\cos\theta=\frac{z+\frac{1}{z}}{2}$$

$$\frac{\varepsilon\cos\theta+1-1}{1+\varepsilon\cos\theta}=1-\frac{1}{1+\varepsilon\cos\theta}$$

$$I=\int_0^1 du\, \frac{u^{\gamma-1}+u^{1-\gamma}-2u}{1-u^2}\cdot 4\pi$$

take up one term

$$\frac{u^{\gamma-1}-u}{\underbrace{1-u^2}_{\text{expand}}}+\frac{u^{1-\gamma}-u}{1-u^2}$$

$$\sum_{k=0}^{\infty} u^{2k}(u^{\gamma-1}-u) = \psi(1)-\psi\left(\frac{\gamma}{2}\right)$$

$$\because\ \psi(x+1)=\psi(1)+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+x}\right) \qquad \psi(1)=-\gamma_E$$

We can get $I = 4\pi x_{01}^{\gamma} \chi(\gamma)$

$\gamma = 1 + 2i\nu = \lambda$

Having this formula, we can easily solve the BFKL eq.

many transformations:

rewrite $T(x_{01}, y) \stackrel{\text{defined}}{=} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} C_\gamma(y)\,(x_{01})^{\gamma}$

$$C_\gamma(y) = \int_0^\infty x_{01}^{-\gamma-1}\, dx_{01}\, T(x_{01}, y)$$

$$\frac{\partial T}{\partial y} = \frac{1}{2\pi i}\int d\gamma\, (x_{01})^{\gamma}\, \frac{\partial C_\gamma(y)}{\partial y}$$

or $$= \frac{1}{2\pi i}\int d\gamma\, C_\gamma(y)\frac{\alpha_s N_c}{2\pi^2}\, 4\pi\, x_{01}^{\gamma}\, \chi(\gamma)$$

$$= \frac{1}{2\pi i}\int d\gamma\, C_\gamma(y)\frac{\alpha_s N_c}{2\pi^2}\left[x_{02}^{\gamma} + x_{12}^{\gamma} - x_{01}^{\gamma}\right]\frac{d^2x_2\, x_{01}^2}{x_{12}^2\, x_{02}^2}$$

(for any γ)

$$\therefore \frac{\partial C_\gamma(y)}{\partial y} = \frac{2\alpha_s N_c}{\pi}\chi(\gamma)\, C_\gamma(y)$$

$$\Rightarrow C_\gamma(y) = C_\gamma(0)\, e^{\frac{2\alpha_s N_c}{\pi}\chi(\gamma)\, y}$$

from $y=0$

substitute into

$$\Rightarrow T(x_{01}, y) = \int_{c-i\infty}^{c+i\infty} x_{01}^{\gamma}\,\frac{1}{2\pi i}\, e^{\frac{2\alpha_s N_c \chi(\gamma)}{\pi} y} \cdot C_\gamma(0)$$

$y = \ln\frac{1}{x}$

From above, we will see:

① Resummation of $(\alpha_s \ln\frac{1}{x})^n$

② Using saddle point approximation to get solution $T(x_{01}, y)$

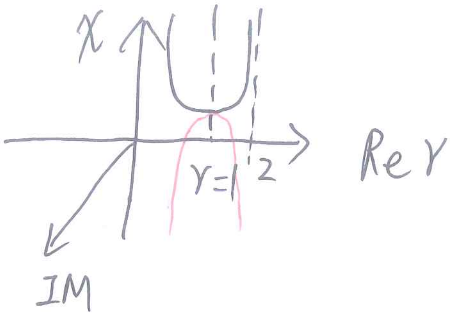
$$T(x_{01}, y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\gamma\, (x_{01})^{\gamma}\, C_{\gamma}(0)\, e^{\frac{2\alpha_s N_c}{\pi}\chi(\gamma) y}$$

complex integral

saddle point at $\gamma = 1$ for $\chi(\gamma)$

$0 < \gamma < 2$

in complex plane.



expand $\chi(\gamma)$ at $\gamma = 1$

high energy limit $\chi(\gamma) = 2\ln 2 + \frac{7}{4}\zeta(3)(\gamma-1)^2 + \cdots$ near $\gamma = 1$

$$T(x_{01}, y) \propto (x_{01} Q_0)^{1} \cdot e^{\frac{4\ln 2\, \alpha_s N_c}{\pi} y - \frac{\ln^2 x_{01} Q_0}{14 \alpha_s N_c \zeta(3) y/\pi}}$$

dimensionless

sometimes, called $\chi(\gamma)$ → BFKL characteristic function.

BFKL dynamics

really large y, $e^{\# y}$ ↑, is not bound state.

BFKL solution is unbounded at very large y.

connection to the solution of DGLAP.

$x_{01} \downarrow \sim \frac{1}{Q \uparrow}$

$$T(x_{01}, y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\gamma\, (x_{01} Q_0)^{\gamma}\, \tilde{C}_{\gamma}(0)\, e^{2\frac{\alpha_s N_c}{\pi}\chi(\gamma) y}$$

put into

$(x_{01} Q_0) << 1$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\gamma\, \tilde{C}_\gamma(0)\, e^{\underbrace{2\frac{\alpha_s N_c}{\pi}\chi(\gamma)y - \frac{\gamma}{2}\ln\frac{1}{x_{01}^2 Q_0^2}}_{\equiv\, h(\gamma)}}$$

$$h'(\gamma)=0$$

saddle point is moved to $\gamma=2$.

$$\chi(\gamma) = \frac{-1}{\gamma-2} + \frac{1}{4}\zeta(3)(\gamma-2)^2$$

$$\Rightarrow T(x_{01},y) \propto (x_{01}Q_0)^2\, e^{2\sqrt{\frac{2\alpha_s N_c}{\pi}\ln\frac{1}{x_{01}^2Q_0^2}\ln\frac{1}{x}}}$$

DGLAP double logarithmic approximation

☆ DGLAP + small-x limit $\Longleftrightarrow$ BFKL + large Q limit

$Q \sim \frac{1}{x_{01}}$

(namely, small x_{01}^2 limit)

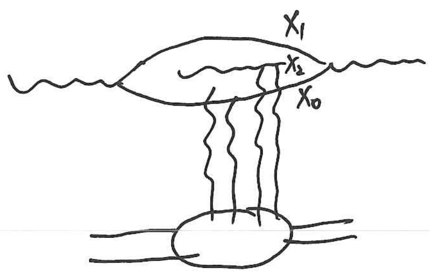
$S = 1 - T$

S: unitary matrix

GBW model

$$T(x_{01},y) = 1 - e^{-\frac{x_{01}^2 Q_0^2}{4}}$$

high energy, gluon distribution $\uparrow$



$T \sim 1$

$T \to 1$, at large y limit, then we should consider the non-linear effect. (Balitsky–Kovchegov (BK) evolution equation)

$$\frac{\partial T(x_{01},y)}{\partial y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2x_2\, \frac{x_{01}^2}{x_{12}^2 x_{02}^2}\left[T(x_{02},y)+T(x_{12},y)-T(x_{01},y) \underbrace{- T(x_{02},y)T(x_{12},y)}_{\text{multiple scattering}}\right]$$

Physics: operator pdf

how define?

roughly speaking, gluon distribution $\downarrow\uparrow \propto T \downarrow\uparrow$

TMD definition: use operator expectation value

borrow TMD concept $\longrightarrow$ in small x TMD $\propto T$

TMD gluon distribution $\Longleftrightarrow$ Small-x gluon distribution

For BK eq, having 2 fixed point

$T=0$ unstable fixed point, deviation from equilibrium

$T=1$ stable fixed point, go back to 1

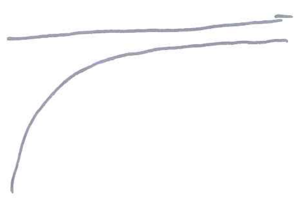
T has a limit

gluon distribution has a limit. gluon distribution is from $T \propto e^{2\sqrt{\frac{2\alpha_s N_c}{\pi}\ln\frac{1}{x_0^2 Q_0^2}\ln\frac{1}{x}}}$

T increases, gluon distribution ~~decrease~~. increases

T ~~unchange~~ ~~decrease~~, gluon distribution ~~unchange~~. decrease.

$T \to 1 \Rightarrow$ saturation.



gluon saturation

gluon distribution has a limit. This fact is not easy to see from σ, but easy to see from unitary consideration.

$SS^\dagger = 1$

partonic picture is clear.

BK eq: From coordinate space to momentum space

rewrite simplified way

paper: S. Munier R. Peschanski 2001 & 2002

scaling solution

traveling wave solution:

In simple language: instead of two variables to a single variable.

$$T \propto T\left(x_{01}^2 Q_s^2(y)\right)$$

$$Q_s^2(y) = Q_0^2 \, e^{\lambda y}$$

initial condition give.

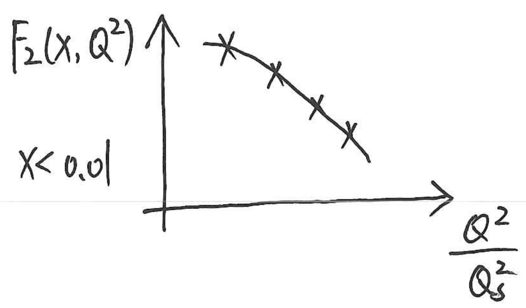
put above expression into F_2

$$F_2(x, Q^2) = \underbrace{\int d^2 b_\perp \int d^2 r_\perp \, |\Psi_{T,L}(r_\perp, Q^2)|^2 \, T(r_\perp, b_\perp)}_{\text{integrate}}$$

$$T \propto T\left(x_{01}^2 Q_s^2(y)\right) \qquad T(r_\perp, y)$$

$$F_2(x, Q^2) = \int d^2 r_\perp \, |\Psi_{T,L}(r_\perp, Q^2)|^2 \, T(r_\perp, y) \propto f\left(\frac{Q^2}{Q_s^2}\right)$$

HERA:



In this plot

$$Q_s^2 = 1\ \text{GeV}^2 \left(\frac{x}{x_0}\right)^{-\lambda}$$

indication of gluon saturation.