Resummation in small-x

small x logarithm $(\alpha_s \ln \frac{1}{x})^n$

physical sudakov logs

small x gluon distribution related to

gluon $\underbrace{\text{TMD distributions.}}_{\text{TMDs}}$ $\Longleftrightarrow$ Small x unintegrated gluon distributions $xG(x, k_\perp)$

TMDs refer to more than 1

different operator definition $\Longrightarrow$ different unintegrated gluon distribution.

evolution: TMDs in general Sudakov logarithm

$\downarrow$

CSS evolution eq.

collions soper Sterman

CSS eq. in some limit $\xrightarrow{?}$ small x evolution eq.

find: CSS eq. is complementary to small-x eq.

reference: arXiv: 1308.2993 (more details)

application:

initially sudakov logarithm : $\alpha_s \ln^2 \frac{Q^2}{k_\perp^2}$ hard scale in scattering

• large logs.

M^2 (Higgs), P_T

$\uparrow$

$S \gg Q^2 \gg k_\perp^2$ kinematics

3 scale problem

$Q^2 \uparrow$, small-x. $\underbrace{\text{TMD evolution}}_{\text{CSS eq.}}$, Sudakov resummation.

build up model.

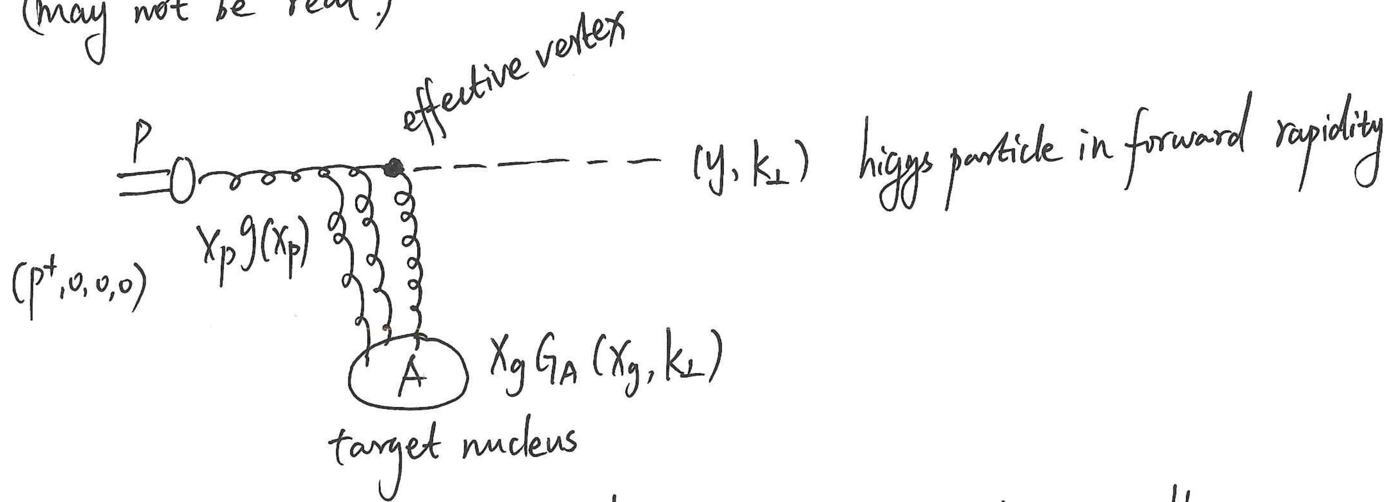
Most simplest higgs production:

$pp \to$ higgs

small x gluon distribution $\xrightarrow{\text{substitute}}$ collinear pdf.

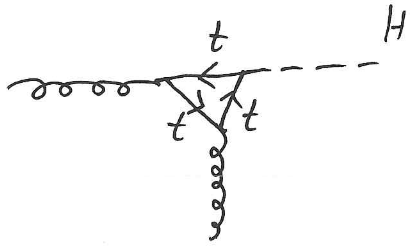
Gedanken experiment

(may not be real)



in principle, top quark is heavy.

$gg \to h(k_\perp, M)$



point particle

$$\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \phi \, g_\phi$$

ϕ ↓ higgs field

g_ϕ → coupling constant

Energy-momentum conservation.

$$x_p = \frac{M_\perp}{\sqrt{s}} e^{y}$$

$$x_g = \frac{M_\perp}{\sqrt{s}} e^{-y}$$

x_p large region $\sim 0.2, 0.3$ collinear region.

x_g small.

collinear pdf don't contain $k_\perp$.

$$M_\perp^2 = \underbrace{k_\perp^2}_{\sim 10\,\text{GeV}} + \underbrace{M^2}_{\sim 126\,\text{GeV}} \approx M^2$$

Simple LO calculation:

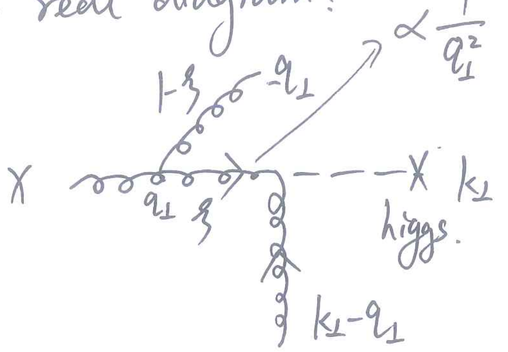
$$\frac{d\sigma}{dy\,d^2k_\perp} = x_p\, g_p(x_p)\, x_g\, G_A(x_g, k_\perp)\, \sigma_0$$

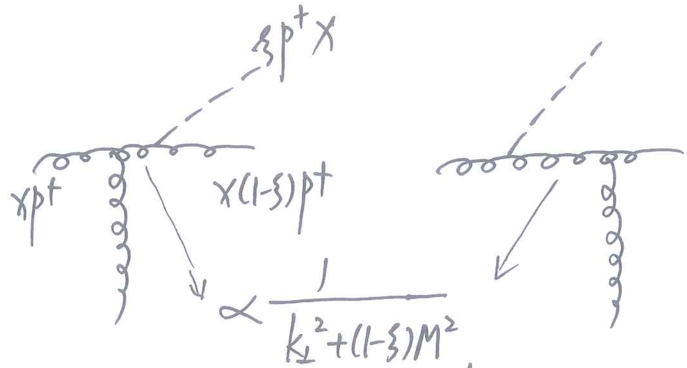
$$\sigma_0 \propto \frac{g_\Phi^2}{g^2}$$

hide in gluon distribution

Consider one loop correction:

3 real diagram:





not power suppressed.

power suppressed unless $(1-\xi)M^2$ is small.

technics:

① power expansion counting

set before $s \gg M^2 \gg k_\perp^2$

look at first diagram

$x > x_p$. $x_p = x\xi$

$$2N_c \int_{x_p}^{1} d\xi\, x\, g(x) \underbrace{\left[\frac{\xi}{1-\xi} + \frac{1-\xi}{\xi} + \xi(1-\xi)\right]}_{\text{splitting function}}$$

splitting function can be expressed in terms of $q_\perp$

$q_\perp$ can be everything, so its integration range can be infinite.

$$\int \frac{d^2q_\perp}{(2\pi)^2} \frac{1}{q_\perp^2}\, x_g'\, G_A(x_g', k_\perp - q_\perp)$$

go to coordinate space

$$\int \frac{d^2 q_\perp}{(2\pi)^2} \frac{1}{q_\perp^2} x_g' G_A(x_g', k_\perp - q_\perp)$$

$$\downarrow$$

$$\int \frac{d^2 x_\perp}{(2\pi)^2} e^{-i(k_\perp - q_\perp) x_\perp} S^{WW}(x_\perp)$$

only have collinear divergence, no UV divergence.

$\xi \to 1$ soft divergence.

② separate collinear region from soft and collinear part

$$\underbrace{2N_c \int_{x_p}^1 d\xi \, x g(x) \left[\frac{\xi}{(1-\xi)_+} + \frac{1-\xi}{\xi} + \xi(1-\xi) \right]}_{\text{collinear div. piece}}$$

$$+ \underbrace{x_p g(x_p) \int_0^1 \frac{2N_c \, d\xi}{1-\xi}}_{\text{collinear div \& soft div.}}$$

$\xi \to 1$ soft div

$q_\perp \to 0$ collinear div.

Using following technics:

dimensional regularization: $\int \frac{d^2 q_\perp}{(2\pi)^2} \frac{1}{q_\perp^2} e^{+i q_\perp x_\perp}$

$$\int \frac{d^2 x_\perp}{(2\pi)^2} e^{-i(k_\perp - q_\perp)} S_A^{WW}(x_p, x_\perp)$$

$\overline{MS}$ scheme:

$$\mu^{2\epsilon} \int \frac{d^{2-2\epsilon} q_\perp}{(2\pi)^{2-2\epsilon}} \frac{1}{q_\perp^2} e^{-i q_\perp x_\perp} = \frac{1}{4\pi} \left[-\frac{1}{\epsilon} + \ln \frac{c_0^2}{\mu^2 x_\perp^2} \right]$$

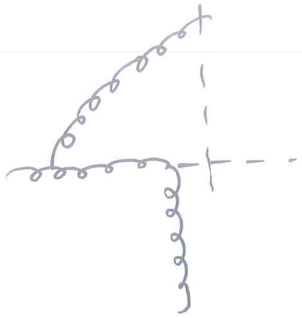
$$\int_0^\infty dx \, e^{-x q_\perp^2} = \frac{1}{q_\perp^2}$$

$$c_0 = 2e^{-\gamma_E}$$

collinear div.

$$x_p\, g(x_p) \int_0^1 \frac{2N_c\, d\xi}{1-\xi}$$

need to look at the kinematics more carefully.



From the kinematics of the leading order graph, we know

$$x_p x_g s = M^2 + k_\perp^2 \approx M^2$$

For real diagram, the one-loop order

$$\text{NLO} \qquad x_p x_g' s = \frac{M^2}{\xi} + \frac{q_\perp^2}{1-\xi} < x_p s$$

when $\xi \to 1$. can forget the first term.

$$1-\xi > \frac{q_\perp^2}{x_p s} \Rightarrow \xi < 1 - \frac{q_\perp^2}{x_p s}$$

$$\int_0^{1-\frac{q_\perp^2}{x_p s}} \frac{d\xi}{1-\xi} = \ln \frac{x_p s}{q_\perp^2} = \ln \frac{x_p s}{M^2} + \ln \frac{M^2}{q_\perp^2}$$

$$= \ln \frac{1}{x_g} + \ln \frac{M^2}{q_\perp^2}$$

small x logarithm

go to small-x evolution,

can derive small-x evolution eq.

$\overline{MS}$ scheme:

$$(4\pi e^{-\gamma_E})^{-\varepsilon} \mu^{2\varepsilon} \int \frac{d^{2-2\varepsilon} q_\perp}{(2\pi)^{2-2\varepsilon}} \frac{1}{q_\perp^2} e^{-iq_\perp \cdot x_\perp} \ln \frac{M^2}{q_\perp^2}$$

collinear div. soft div.

soft & collinear div.

$$\left(\frac{M^2}{q_\perp^2}\right)^a = e^{a\ln\frac{M^2}{q_\perp^2}}$$

derivative to a, then let $a \to 0$.

$$\frac{\partial e^{a\ln\frac{M^2}{q_\perp^2}}}{\partial a} = \ln\frac{M^2}{q_\perp^2}\, e^{a\ln\frac{M^2}{q_\perp^2}}$$ when $a \to 0$, get $\ln\frac{M^2}{q_\perp^2}$

very useful trick.

another trick:

$$\Gamma(a)\,(q_\perp^2)^{-a} = \int_0^\infty dx\, x^{a-1} e^{-x q_\perp^2}$$

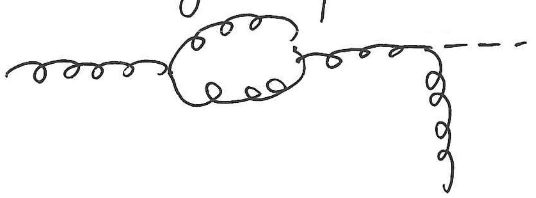
at last, real contribution:

$$(4\pi e^{-\gamma_E})^{-\varepsilon} \mu^{2\varepsilon} \int \frac{d^{2-2\varepsilon} q_\perp}{(2\pi)^{2-2\varepsilon}} \frac{1}{q_\perp^2} \ln\frac{M^2}{q_\perp^2} e^{-iq_\perp\cdot x_\perp}$$

$$= \frac{1}{4\pi}\left[\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\frac{M^2}{\mu^2} + \frac{1}{2}\left(\ln\frac{M^2}{\mu^2}\right)^2 - \frac{1}{2}\left(\ln\frac{M^2 x_\perp^2}{c_0^2}\right)^2 - \frac{\pi^2}{12}\right]$$

3 virtual diagram:

gluon loop



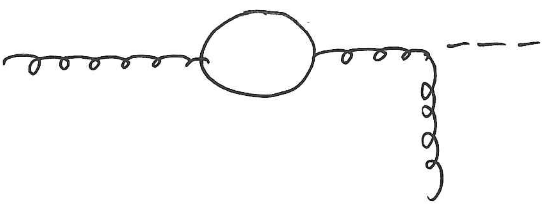
$\propto \quad \mu^{2\varepsilon}\int \frac{d^{2-2\varepsilon} q_\perp}{(2\pi)^{2-2\varepsilon}} \frac{1}{q_\perp^2} \equiv 0$ by definition

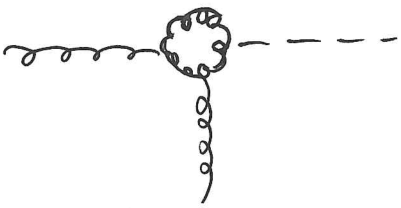
uv & IR div.

in practice $\quad \frac{1}{4\pi}\left[-\frac{1}{\varepsilon_{IR}} + \ln\frac{Q^2}{\mu^2} + \boxed{\frac{1}{\varepsilon_{UV}} - \ln\frac{Q^2}{\mu^2}}\right]$

finite.

$\alpha_s(Q^2)$





$$\propto \frac{1}{4\pi}\left[-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln\frac{M^2}{\mu^2} - \frac{1}{2}\ln^2\frac{M^2}{\mu^2} + \frac{\pi}{2} + \frac{\pi^2}{12}\right]$$

universal

depend on gauge.

add the real and virtual contribution

$$\text{real} + \text{virtual} = \frac{1}{4\pi}\left[-\frac{1}{2}\ln^2\frac{M^2 X_\perp^2}{C_0^2} + \frac{\pi^2}{2}\right]$$

$C_0 = 2e^{-\gamma_E}$ CSS resummation

Full DGLAP eq.

$$\underset{\text{real}}{\cdots\cdots} + 2N_c\,\delta(1-\xi)\underbrace{\left[\frac{11}{12} - \frac{2n_f/N_c}{12}\right]}_{\beta_0}$$ first two virtual diagrams contribute

$$\beta_0 = \frac{11}{12} - \frac{n_f}{6N_c}$$

at last, divergence cancle.

$$\frac{d\sigma}{dy\,d^2k_\perp} = \int\frac{d^2X_\perp}{(2\pi)^2}e^{-ik_\perp\cdot X_\perp}\,S^{WW}(x_g, X_\perp)\cdot$$

(one-loop)

$$\left\{1 + \cancel{4}\alpha_s N_c\cdot\frac{1}{\cancel{4}\pi}\left[-\frac{1}{2}\ln^2\frac{M^2X_\perp^2}{C_0^2} + \frac{\pi^2}{2} + \beta_0\underbrace{\ln\frac{Q^2}{\mu^2}}_{\beta_0\ln\frac{M^2X_\perp^2}{C_0^2}}\right]\right\}$$

$$\cdot\, x_p\, g\left(x_p, \frac{C_0^2}{X_\perp^2}\right)$$

choose $\mu^2 = \frac{C_0^2}{X_\perp^2}$

choose $Q^2 = M^2$

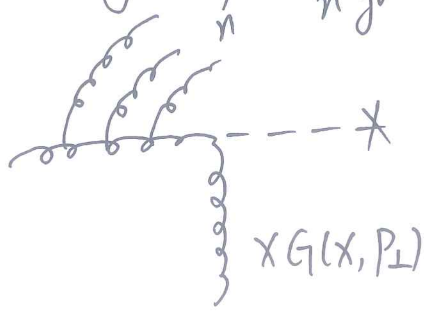
one-loop result:

useful trick:

③ either use CSS evolution or resummation recover one-loop.

resummation is very simple: n gluons radiation n final gluons looks same.

assume



$$\sum_n \frac{1}{n!} \int d^2k_1\, d^2k_2 \cdots d^2k_n\, S(k_1) S(k_2) S(k_3) \cdots S(k_n)$$

$$\cdot \int d^2P_\perp\, xG(x, P_\perp)\, \underbrace{(2\pi)^2 \delta^{(2)}(k_1 + k_2 + \cdots + k_n + P_\perp - k_\perp)}_{\int d^2b_\perp\, e^{ib_\perp (k_1 + k_2 + \cdots + k_n + P_\perp - k_\perp)}}$$

$$= \sum_n \frac{1}{n!} \int d^2b_\perp\, \tilde{S}^n(b_\perp) S^{WW}(b_\perp) e^{-ik_\perp \cdot b_\perp}$$

have done fourier transform

1 loop → n loop

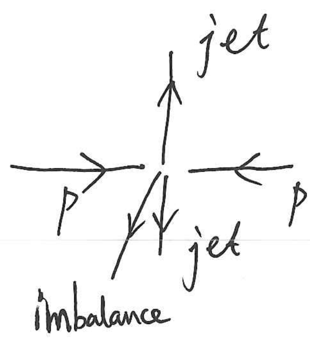
$$\frac{d\sigma}{dy\, d^2k_\perp} = \sigma_0 \int \frac{d^2x_\perp}{(2\pi)^2} e^{-ik_\perp x_\perp} e^{-A \ln^2 \frac{M^2 x_\perp^2}{c_0^2} + B \ln \frac{M^2 x_\perp^2}{c_0^2}} (1+C)$$

$$\cdot\, x_p g\left(x_p, \frac{c_0^2}{x_\perp^2}\right) S^{WW}(x_g, x_\perp)$$

a double logarithm, Sudakov resummation.

three scales:

as long as $s \uparrow$, $Q^2 \uparrow$, measure finite $k_\perp^2$ → Sudakov resummation.



mid-y

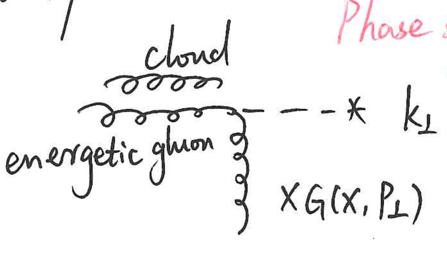
$P_J\, \Delta\phi$

di-jet production

di-jet angle distribution.

interpretation: Sudakov resummation = Sudakov suppression

Phase space constrait, Suppressed by Sudakov factor.



A → double logarithm

B → single logarithm, B is positive at least for gluon.

C → $\frac{\alpha_s}{\pi} \cdot \frac{\pi^2}{2} N_c$