# QCD Running coupling

by Lin Yan

QCD Lagrangian:

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{ferm} + \mathcal{L}_{gf} + \mathcal{L}_{FP}$$

Since we only interested in the QCD running coupling, we just list the relevant Feynmen rules in here.

$$\mathcal{L}_0 = \bar{\psi}_0 i \not{\partial} \psi_0 + g_{s0} \bar{\psi}_0 \gamma^\mu T^a \psi_0 A^a_\mu - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

Redefine the fields and parameters.

$$A^0_\mu = Z_A^{1/2} A_\mu \quad , \quad \psi_0 = Z_\psi^{1/2} \psi \;, \quad g_s^0 = Z_g g_s$$

$$\mathcal{L} = \mathcal{L}_r + \delta\mathcal{L}$$

$$= Z_\psi \bar{\psi} i \not{\partial} \psi + g Z_g Z_\psi Z_A^{1/2} \bar{\psi} \gamma^\mu T^a \psi A^a_\mu - \frac{1}{4} Z_A F^a_{\mu\nu} F^{\mu\nu a} + \cdots$$

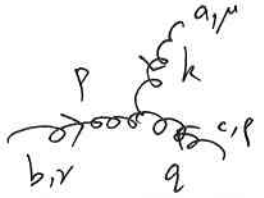
write $Z_i = 1 + \delta Z_i$ , define $Z_F = Z_g Z_\psi Z_A^{1/2}$

$$\mathcal{L}_r = \bar{\psi} i \not{\partial} \psi + g \bar{\psi} \gamma^\mu T^a \psi A^a_\mu - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

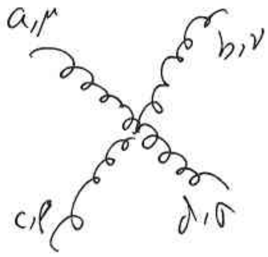
$$\delta\mathcal{L} = \delta_\psi \bar{\psi} i \not{\partial} \psi + \delta Z_F g \bar{\psi} \gamma^\mu T^a \psi A^a_\mu - \frac{1}{4} \delta Z_A F^a_{\mu\nu} F^{\mu\nu a} + \cdots$$

Feynman rules:


$$ig_s\gamma^\mu T^a$$



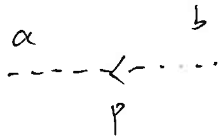

$$gf^{abc}[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu]$$




$$\begin{aligned}-ig^2[\, & f^{ade}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}) \\ + & f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}) \\ + & f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})]\end{aligned}$$


$$-gf^{abc}p^\mu$$

Ghost propagator:

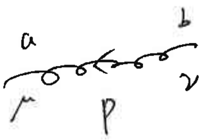



$$\frac{i\delta^{ab}}{p^2+i\epsilon}$$

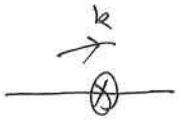
gluon propagator

$$\frac{-i}{p^2}\delta_{ab}\left[g_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{p^2}\right]$$

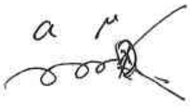
$\xi=1$, Feynman-'t Hooft gauge



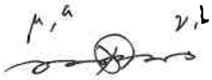

$$= \frac{-i}{p^2}\delta_{ab}\, g_{\mu\nu}$$



$i \not{k}\, \delta Z_4$



$i \delta Z_F\, g \gamma^\mu T^a$



$-i (k^2 g^{\mu\nu} - k^\mu k^\nu) \delta^{ab} \delta Z_A$

From the definition, we see

$$g_s = \frac{g_{s0}}{Z_g} = g_{s0} \frac{Z_4 Z_A^{1/2}}{Z_F}$$

Therefore, we should consider ~~three~~ class graphs which are showed as follows:

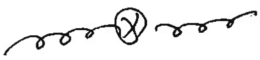
① The vacuum polarization graphs

$(a_1)$

$(a_2)$

$(a_3)$

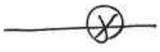
$(a_4)$



$(a_5)$

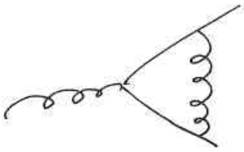
(b) The self energy of fermion

(b1)

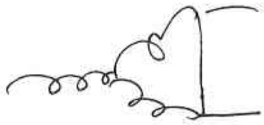


(b2)

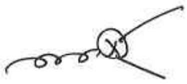
(c) The vertex graphs



(C1)



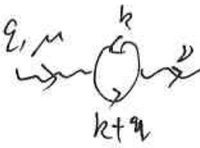
(C2)



(C3)

G: The vaccum polarization with fermion graph.

 $= \mathrm{Tr}\,[T^a T^b] \otimes$ 



$$= -\int \frac{d^n k}{(2\pi)^n} (i g_s \mu^\epsilon)^2 \, \mathrm{Tr}\left[\gamma^\mu \frac{i}{\not k} \gamma^\nu \frac{i}{\not k + \not q}\right]$$

$$= -g_s^2 \mu^{2\epsilon} \int \frac{d^n k}{(2\pi)^n} \frac{\mathrm{Tr}[\gamma^\mu \not k \gamma^\nu (\not k + \not q)]}{k^2 (k+q)^2}$$

$$\mathrm{Tr}[\gamma^\mu \not k \gamma^\nu (\not k + \not q)] = -4k^2 g^{\mu\nu} - 4 g^{\mu\nu}(k\cdot q) + 8k^\mu k^\nu + 4k^\nu q^\mu + 4k^\mu q^\nu \quad (1)$$

$$\frac{1}{k^2 (k+q)^2} = \int_0^1 dx \frac{1}{[x(k+q)^2 + (1-x)k^2]^2} = \int_0^1 dx \frac{1}{[k^2 + 2k\cdot q x + q^2 x]^2}$$

$$= \int_0^1 dx \frac{1}{[l^2 - \Delta]^2}$$

where $l = k + xq$ , $\Delta = x(x-1) q^2$

$$\mathcal{D} = -4(l-xq)^2 g^{\mu\nu} - 4g^{\mu\nu}(l-xq)\cdot q + 8(l-xq)^\mu (l-xq)^\nu$$
$$+4(l-xq)^\nu q^\mu + 4(l-xq)^\mu q^\nu$$

terms linear in $l$ vanish after integrate out momentum $l$.

$$\Rightarrow -4[l^2 g^{\mu\nu} + x^2 q^2 g^{\mu\nu} + g^{\mu\nu}(-xq)\cdot q - 2l^\mu l^\nu - 2x^2 q^\mu q^\nu + 2xq^\nu q^\mu] + \text{linear in } l$$

$$= 4[2l^\mu l^\nu - g^{\mu\nu} l^2 - 2x(1-x) q^\mu q^\nu + g^{\mu\nu} x(1-x) q^2] + \text{linear in } l$$

note: $$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \quad (a)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \quad (b)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2-\Delta)^n} = \frac{(-1)^{n-1}}{(4\pi)^{d/2}} i \frac{g^{\mu\nu}}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \quad (c)$$

For our case:

$$(a) \Rightarrow \frac{(-1)^2 i}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(2-\frac{n}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{n}{2}}$$

$$n = 4-2\epsilon$$
$$\Gamma(2-\tfrac{n}{2}) = \Gamma(\epsilon)$$
$$\Gamma(1-\tfrac{n}{2}) = \Gamma(\epsilon-1) = \frac{\Gamma(\epsilon)}{\epsilon-1}$$

$$(b) \Rightarrow \frac{(-1)^{\cancel{\#}} i}{(4\pi)^{\frac{n}{2}}} \frac{n}{2} \frac{\Gamma(1-\frac{n}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{-\frac{n}{2}}$$

$$(c) \Rightarrow \frac{(-1)}{(4\pi)^{\frac{n}{2}}} \frac{i g^{\mu\nu}}{2} \frac{\Gamma(1-\frac{n}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{1-\frac{n}{2}}$$

After integrate out the momentum.

$$\Rightarrow \frac{2(-1)i}{(4\pi)^{\frac{n}{2}}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(\epsilon-1)}{1} \left(\frac{1}{\Delta}\right)^{\epsilon} x(x-1) q^2 - g^{\mu\nu} \frac{(-1)i}{(4\pi)^{\frac{n}{2}}} \frac{n}{2} \Gamma(\epsilon-1) \left(\frac{1}{\Delta}\right)^{\epsilon} x(x-1) q^2$$

$$+ [-2x(1-x) q^\mu q^\nu + g^{\mu\nu} x(1-x) q^2] \frac{(-1)^2 i}{(4\pi)^{\frac{n}{2}}} \Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^{\epsilon}$$

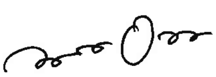
$$= \left[2 g^{\mu\nu} x(1-x) q^2 - 2x(1-x) q^\mu q^\nu\right] \frac{i}{(4\pi)^{\frac{d}{2}}} \Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^\epsilon$$

$$= \left[q^2 g^{\mu\nu} - q^\mu q^\nu\right] i \, \frac{2x(1-x)}{(4\pi)^{\frac{d}{2}}} \Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^\epsilon$$

$$= Tr[T^a T^b] \, i \left[q^2 g^{\mu\nu} - q^\mu q^\nu\right] \left\{ -\frac{8 g_s^2 \mu^{2\epsilon}}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \, x(1-x) \, \Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^\epsilon \right\}$$

note $\Delta = x(x-1) q^2$

$Tr[T^a T^b] = C(r) \delta^{ab}$

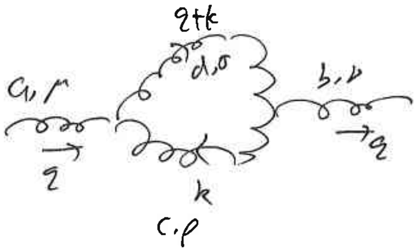


$$= C(r) \delta^{ab} \, i \left[q^2 g^{\mu\nu} - q^\mu q^\nu\right] \frac{-8 g_s^2 \mu^{2\epsilon}}{(4\pi)^2} \frac{1}{(4\pi)^{-\epsilon}} \left\{ \int_0^1 dx \, x(1-x) \, \Gamma(\epsilon) \left[\frac{1}{x(x-1) q^2}\right]^\epsilon \right\}$$

$$= i \left[q^2 g^{\mu\nu} - q^\mu q^\nu\right] \delta^{ab} \left[ -\frac{g_s^2}{(4\pi)^2} \left[\frac{4\pi\mu^2}{q^2}\right]^\epsilon C(r) \frac{4}{3} \Gamma(\epsilon) \right] + \ldots$$

we only focus on the div part

$$= i \left[q^2 g^{\mu\nu} - q^\mu q^\nu\right] \delta^{ab} \left[ -\frac{g_s^2}{(4\pi)^2} \frac{4}{3} C(r) \left(\frac{4\pi\mu^2}{q^2}\right)^\epsilon \Gamma(\epsilon) \right]$$




$$= \frac{1}{2}\int \frac{d^n k}{(2\pi)^n} (-g_s \mu^\epsilon)^2 f^{acd} \left[ g^{\mu\rho}(q-k)^\sigma + g^{\rho\sigma}(2k+q)^\mu + g^{\sigma\mu}(-2q-k)^\rho \right]$$

$$\frac{-i}{k^2} \quad f^{cbd}\left[ \delta^\nu_\rho (-k+q)_\sigma + \delta^\nu_\sigma (-q-q-k)_\rho + g_{\rho\sigma}(q+k+q)_\nu \right]$$

$$\frac{-i}{(q+k)^2}$$

$$= \frac{1}{2}\int \frac{d^n k}{(2\pi)^n} g_s^2 \mu^{2\epsilon} f^{acd} f^{bcd} \frac{-i}{k^2} \frac{-i}{(q+k)^2} N^{\mu\nu}$$

from symm. factor

$$N^{\mu\nu} = \left[ g^{\mu\rho}(q-k)^\sigma + g^{\rho\sigma}(2k+q)^\mu + g^{\sigma\mu}(-2q+k)^\rho \right]$$
$$\left[ \delta^\nu_\rho (k-q)_\sigma - g_{\rho\sigma}(2q+k) + \delta^\nu_\sigma (2q+k)_\rho \right]$$

Note: $f^{acd} f^{bcd} = C_2(G)\delta^{ab}$

$$\frac{1}{k^2}\frac{1}{(k+q)^2} = \int_0^1 dx \frac{1}{(l^2-\Delta)^2} \qquad \text{with } l = k + xq, \quad \Delta = x(x-1)q^2$$

$$N^{\mu\nu} \Rightarrow -g^{\mu\nu} l^2\, 6\left(1-\frac{1}{n}\right) - g^{\mu\nu} q^2\left[(2-x)^2 + (1+x)^2\right]$$
$$+ q^\mu q^\nu \left[ (2-n)(1-2x)^2 + 2(1+x)(2-x) \right]$$

$$\int \frac{d^n k}{(2\pi)^n} \frac{1}{(l^2-\Delta)^2} N^{\mu\nu}$$

$$= \left[-g^{\mu\nu}\, 6\left(1-\frac{1}{n}\right)\right] \frac{(-1)\, i}{(4\pi)^{\frac{n}{2}}} \frac{n}{2} \frac{\Gamma(1-\frac{n}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{n}{2}} x(x-1)q^2$$

$$-g^{\mu\nu} q^2 \left[(2-x)^2 + (1+x)^2\right] \frac{(-1)^2\, i}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(2-\frac{n}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{n}{2}}$$

$$+q^\mu q^\nu \left[(2-n)(1-2x)^2 + 2(1+x)(2-x)\right] \frac{(-1)^2}{(4\pi)^{\frac{n}{2}}}\, i\, \frac{\Gamma(2-\frac{n}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{n}{2}}$$

$$\xrightarrow{n\to 4} i \frac{9}{(4\pi)^2}(-g^{\mu\nu})\, \Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^\epsilon x(x-1)q^2$$

$$- i \frac{1}{(4\pi)^2} g^{\mu\nu}\, \Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^\epsilon \left[(2-x)^2+(1+x)^2\right] q^2$$

$$+ i \frac{1}{(4\pi)^2} q^\mu q^\nu\, \Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^\epsilon \left[-2(1-2x)^2 + 2(1+x)(2-x)\right]$$

$$= (-11x^2+11x-5) \frac{i}{(4\pi)^2} (4\pi)^\epsilon g^{\mu\nu}\, \Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^\epsilon q^2$$

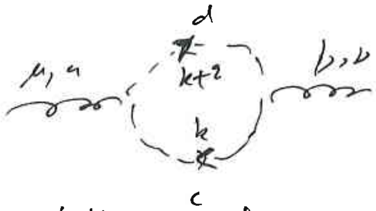
$$(-10x^2+10x+2) \frac{i\,(4\pi)^\epsilon}{(4\pi)^2} q^\mu q^\nu\, \Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^\epsilon$$

$$= -\frac{g_s^2 \mu^{2\epsilon}}{2\times(4\pi)^2} C_2(G)\, \delta^{ab} (4\pi)^\epsilon \int_0^1 dx \left(\frac{1}{\Delta}\right)^\epsilon \Gamma(\epsilon)$$

$$\left\{(-11x^2+11x-5) g^{\mu\nu} q^2 + (-10x^2+10x+2) q^\mu q^\nu\right\}$$

$$= \frac{g_s^2}{(4\pi)^2} C_2(G)\, \delta^{ab} \left(\frac{4\pi\mu^2}{q^2}\right)^\epsilon \Gamma(\epsilon) \int_0^1 dx \left[\frac{1}{x(x-1)}\right]^\epsilon$$

$$\left\{\frac{(11x^2-11x+5)}{2} g^{\mu\nu} q^2 + \frac{10x^2-10x-2}{2} q^\mu q^\nu\right\}$$




ghost field anticommut..

$$= -\int \frac{d^nk}{(2\pi)^n}\, g_s \mu^\epsilon f^{dca} (k+q)^\mu \frac{i}{k^2}\, g_s \mu^\epsilon f^{cdb} k^\nu \frac{i}{(k+q)^2}$$

$$= -\int \frac{d^nk}{(2\pi)^n}\, g_s^2 \mu^{2\epsilon} f^{dac} f^{cbd} \frac{i}{k^2}\frac{i}{(k+q)^2}(k+q)^\mu k^\nu$$

$$f^{dac}f^{cbd} = -C_2(G)\,\delta^{ab}$$

$$\Rightarrow\ -C_2(G)\,\delta^{ab} g_s^2\mu^{2\epsilon}\int\frac{d^nk}{(2\pi)^n}\frac{(k+q)^\mu k^\nu}{k^2(k+q)^2}$$

$$\frac{1}{k^2(k+q)^2} = \int_0^1 dx\, \frac{1}{(l^2-\Delta)^2}$$ with $\Delta = x(x-1)q^2$, $l = k+xq$

$$(k+q)^\mu k^\nu \Rightarrow l^\mu l^\nu + x(x-1)q^\mu q^\nu$$

$$\Rightarrow\ \frac{(-1)i}{(4\pi)^{\frac n2}}\frac{g^{\mu\nu}}{2}\frac{\Gamma(1-\frac n2)}{\Gamma(2)}\left(\frac1\Delta\right)^{2-\frac n2} x(x-1)q^2$$

$$+\ x(x-1)q^\mu q^\nu \frac{(-1)^2\, i}{(4\pi)^{\frac n2}}\frac{\Gamma(2-\frac n2)}{\Gamma(2)}\left(\frac1\Delta\right)^{2-\frac n2}$$

$$= \Gamma(\epsilon)\left(\frac1\Delta\right)^\epsilon \frac{i}{(4\pi)^{2-\epsilon}}\left[\frac{g^{\mu\nu}}{2}x(x-1)q^2 + (x-1)x\,q^\mu q^\nu\right]$$

$$= \frac{i g_s^2\mu^{2\epsilon}}{(4\pi)^{2-\epsilon}}\, C_2(G)\,\delta^{ab}\int_0^1 dx \left(\frac1\Delta\right)^\epsilon\Gamma(\epsilon)$$
$$\times\left[-\frac{g^{\mu\nu}}{2}x(x-1)q^2 - (x-1)x\,q^\mu q^\nu\right]$$

$$= \frac{i g_s^2}{(4\pi)^2}\, C_2(G)\,\delta^{ab}\left[\frac{4\pi\mu^2}{q^2}\right]^\epsilon\int_0^1 dx\left[\frac{1}{x(x-1)}\right]^\epsilon\Gamma(\epsilon)$$
$$\times\left[\frac{g^{\mu\nu}}{2}(1-x)x\,q^2 + (1-x)x\,q^\mu q^\nu\right]$$


$$= \frac{1}{2}\int \frac{d^n k}{(2\pi)^n} \frac{-i g_{\rho\sigma}}{k^2} \delta^{cd} \left[-i g_s^2 \mu^{2\epsilon}\right]$$

$$\big[-f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \longrightarrow \text{antisymmetry } \rho \leftrightarrow \sigma$$

$$+ f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho})$$

$$+ f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})\big]$$

$$f^{ace} f^{bde} \delta^{cd} = f^{ace} f^{bce} = C_2(G)\,\delta^{ab}$$

$$f^{ade} f^{bce} \delta^{cd} = C_2(G)\,\delta^{ab}$$

$$= -\frac{1}{2} C_2(G)\,\delta^{ab} \int \frac{d^n k}{(2\pi)^n} \frac{g_s^2 \mu^{2\epsilon}}{k^2} g^{\mu\nu} (n-1) \times 2$$

$$= -g_s^2 \mu^{2\epsilon} C_2(G)\,\delta^{ab} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2} g^{\mu\nu} (n-1)$$

$$\int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2} = \int \frac{d^n k}{(2\pi)^n} \frac{(k+q)^2}{k^2 (k+q)^2} = \int \frac{d^n l}{(2\pi)^n} \int_0^1 dx \frac{l^2 + x^2 q^2}{(l^2 - \Delta)^2}$$

$$= \frac{i}{(4\pi)^{2-\epsilon}} \int_0^1 dx \left[\Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^{\epsilon} 2x(x-1) q^2 + x^2 q^2 \Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^{\epsilon}\right]$$

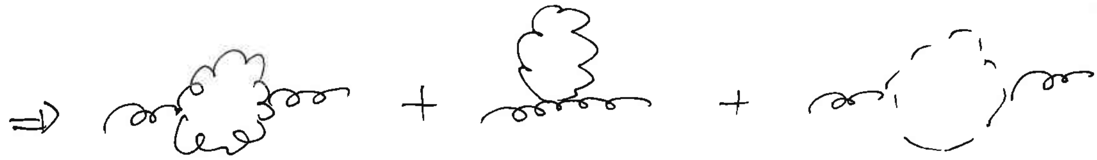
$$\Rightarrow \frac{i g_s^2 \mu^{2\epsilon}}{(4\pi)^{2-\epsilon}} C_2(G)\,\delta^{ab} \int_0^1 dx\, \Gamma(\epsilon) \left(\frac{1}{\Delta}\right)^{\epsilon} 3 g^{\mu\nu} q^2 (3x^2 - 2x)$$

note: $\int 3x^2 - 2x = 0$,

it don't contribute to the $\beta$ function

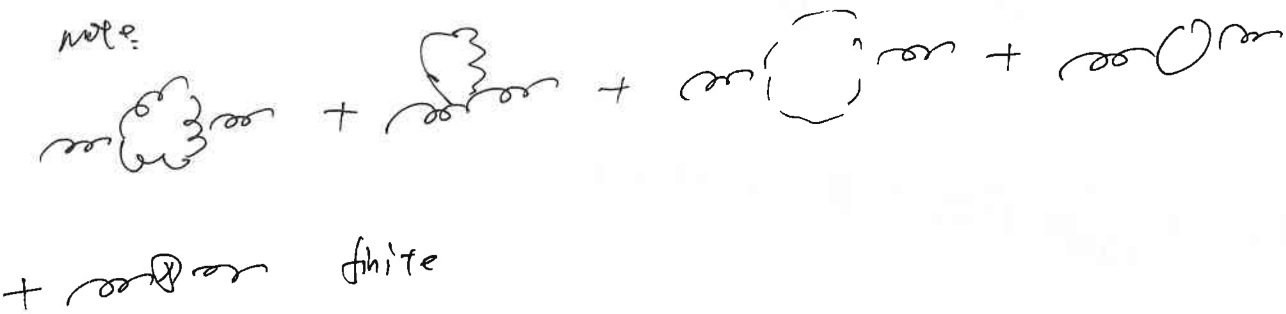
$$\int_0^1 dx \left[ \frac{11x^2-11x+5}{2} + \frac{(1-x)x}{2} \right] = \frac{5}{3}$$

$$\int_0^1 dx \left[ \frac{12x^2-12x-2}{2} + (1-x)x \right] = \frac{5}{-3}$$



$$= i(q^2 g^{\mu\nu} - q^\mu q^\nu)\,\delta^{ab}\,\frac{g_s^2}{(4\pi)^2}\,\frac{5}{3}\,C_2(G)\left(\frac{4\pi\mu^2}{q^2}\right)^{\epsilon}\Gamma(\epsilon)$$

note:

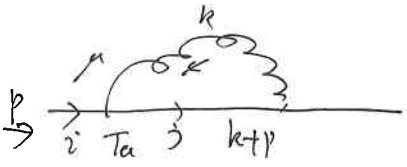


finite

For the $\overline{MS}$ scheme:

$$\delta Z_A = \frac{g_s^2}{16\pi^2}\left[\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\right]\left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right)$$

## The Fermion self energy




$$= \int \frac{d^nk}{(2\pi)^n}(i g_s \mu^\epsilon)^2\, T^a \gamma_\mu \frac{i(\not p+\not k)}{(p+k)^2}\gamma^\mu T^a \frac{-i}{k^2}$$

$$= -(T^aT^a)\, g_s^2 \mu^{2\epsilon}\int \frac{d^nk}{(2\pi)^n}\frac{\gamma_\mu(\not p+\not k)\gamma^\mu}{k^2(p+k)^2}$$

$$\gamma_\mu(\not p+\not k)\gamma^\mu = -(n-2)(\not p+\not k)$$

 $= C_2(r)\, g^2 (n-2) \int \frac{d^n k}{(2\pi)^n} \frac{\not p + \not k}{k^2 (p+k)^2}$

$$\frac{1}{k^2 (p+k)^2} = \int_0^1 dx \frac{1}{(l^2-\Delta)^2} \quad \text{with } \Delta = x(x-1)p^2 \,, \quad l = k + xp$$

$$\int \frac{d^n k}{(2\pi)^n} \frac{\not p + \not k}{k^2 (k^2+p^2)}$$

$$= \int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \frac{(1-x)\not p}{(l^2-\Delta)^2} = \int_0^1 dx\, (1-x)\not p \, \frac{(-1)^2\, i}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(2-\frac{n}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{n}{2}}$$

$$= \int_0^1 dx\, (1-x)\not p \, \frac{i}{(4\pi)^{2-\epsilon}} \Gamma(\epsilon) \frac{1}{x(x-1)} \left(\frac{1}{q^2}\right)^{\epsilon}$$

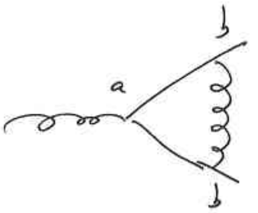
$\boxed{q = p}$

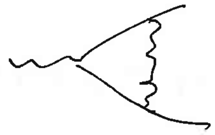
$\Rightarrow$  $= \frac{i g^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{q^2}\right)^{\epsilon} \not p \; C_2(r)\, \Gamma(\epsilon)$

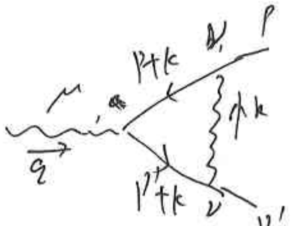
 $+$  $=$ finite

For $\overline{MS}$ scheme:

$$\delta Z_4 = -\frac{g^2}{16\pi^2} C_2(r) \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right)$$




$$= T^b T^a T^b \otimes$$ 




$$= \int \frac{d^nk}{(2\pi)^n} (i g_s \mu^\epsilon)\gamma_\nu \frac{i(\not p' + \not k)}{(p'+k)^2} \, i g_s \mu^\epsilon \gamma_\mu \frac{i(\not p+\not k)}{(p+k)^2}(i g_s\mu^\epsilon)\gamma^\nu \frac{-i}{k^2}$$

$$= g_s\mu^\epsilon \, g_s^2\mu^{2\epsilon}\int \frac{d^nk}{(2\pi)^n}\frac{\gamma_\nu(\not p'+\not k)\gamma_\mu(\not p+\not k)\gamma^\nu}{(p'+k)^2(p+k)^2k^2}$$

Since we only interested in the divergence part, so

$$\int \frac{d^nk}{(2\pi)^n}\frac{\gamma_\nu(\not p'+\not k)\gamma_\mu(\not p+\not k)\gamma^\nu}{(p'+k)^2(p+k)^2k^2} \Rightarrow \int \frac{d^nk}{(2\pi)^n}\frac{\gamma_\nu \not k\gamma_\mu \not k\gamma^\nu}{k^2k^2k^2}$$

$$\gamma_\nu\not k\gamma_\mu\not k\gamma^\nu = -(n-2)(2k^\mu\not k - k^2\gamma^\mu).$$

$$k^\mu k^\alpha \to \frac{g^{\mu\alpha}k^2}{n}$$

$$\Rightarrow k^2\gamma^\mu$$

$$= g_s\mu^\epsilon \, g_s^2\mu^{2\epsilon}\gamma_\mu\int\frac{d^nk}{(2\pi)^n}\frac{1}{(k^2)^2}$$

$$= g_s\mu^\epsilon \, g_s^2\mu^{2\epsilon}\gamma_\mu \frac{(-1)}{(4\pi)^{\frac{n}{2}}} \, i\,\Gamma(\epsilon)\left(\frac{1}{q^2}\right)^\epsilon$$

$$= g_s\mu^\epsilon \, g_s^2\gamma_\mu\frac{i}{(4\pi)^2}\left(\frac{4\pi\mu^2}{q^2}\right)^\epsilon\Gamma(\epsilon)$$

$$[T^a, T^b] = i f^{abc}T^c$$

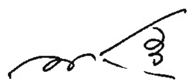
$$T^bT^aT^b = T^bT^aT^b + T^b[T^a, T^b]$$

$$= C_2(r)T^a + T^b i f^{abc}T^c$$

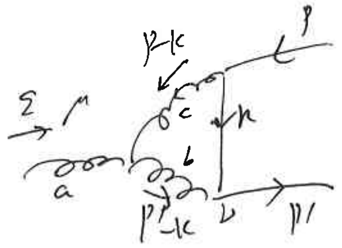
$$= C_2(r)T^a + \frac{i}{2}[T^bT^cf^{abc} + T^bT^cf^{abc}]$$

$$= C_2(r)T^a + \frac{i}{2}[T^b, T^c]f^{abc}$$

$$= [C_2(r) - \frac{1}{2}C_2(G)]\,T^a$$



$$= i g_s\mu^\epsilon \frac{g_s^2}{16\pi^2}[C_2(r) - \frac{1}{2}C_2(G)]\,T^a\gamma^\mu \left(\frac{4\pi\mu^2}{q^2}\right)^\epsilon \Gamma(\epsilon)$$



$$= \int \frac{d^nk}{(2\pi)^n}(i g_s\mu^\epsilon\gamma_\nu T^b)\frac{i}{\not k}(i g_s\gamma_\rho T^c)\frac{-i}{(p-k)^2}$$

$$g_s\mu^\epsilon f^{abc}[g^{\mu\nu}(p'-p+p-k)^\rho + g^{\nu\rho}(-p'+k-p+k)^\mu + g^{\rho\mu}(p-k-p'+p)^\nu]\frac{i}{(p'-k)^2}]$$

$$= \int \frac{d^nk}{(2\pi)^n} i g_s\mu^\epsilon\gamma_\nu T^b \frac{i\not k}{k^2} i g_s\mu^\epsilon\gamma_\rho T^c \frac{-i}{(p-k)^2}\frac{-i}{(p'-k)^2}$$

$$g_s\mu^\epsilon f^{abc}[g^{\mu\nu}(2p'-k-p)^\rho + g^{\nu\rho}(-p'-p+2k)^\mu + g^{\rho\mu}(2p-k-p')^\nu]$$

$$f^{abc}T^bT^c = \frac{1}{2}[f^{abc}T^bT^c + f^{abc}T^bT^c]$$

$$= \frac{1}{2}f^{abc}[T^b, T^c] = \frac{i}{2}C_2(G)\delta^{ab}T^a$$

$$= -\int\frac{d^nk}{(2\pi)^n}\frac{g_s^3\mu^{3\epsilon}}{2}C_2(G)\,T^a\,\frac{\gamma_\nu\not k\gamma_\rho[g^{\mu\nu}(2p'-k-p)^\rho + g^{\nu\rho}(-p'-p+2k)^\mu + g^{\rho\mu}(2p-k-p')^\nu]}{k^2(p-k)^2(p'-k)^2}$$

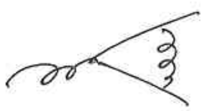
only focus on the uv part.

$$= -\int \frac{d^n k}{(2\pi)^n} \frac{g_s^3 \mu^{3\epsilon}}{2} C_2(G)\, T^a \, \gamma_\nu \not{k} \gamma_\rho \frac{[g^{\mu\nu}k^\rho - 2g^{\nu\rho}k^\mu + g^{\rho\mu}k^\nu]}{[k^2]^3}$$

$$\gamma_\nu \not{k} \gamma_\rho [g^{\mu\nu}k^\rho - 2g^{\nu\rho}k^\mu + g^{\rho\mu}k^\nu] \Rightarrow 3\gamma^\mu k^2$$

$$= \frac{i g_s \mu^\epsilon}{(4\pi)^{2-\epsilon}} \frac{3}{2} C_2(G)\, g_s^2 \mu^{2\epsilon} T^a \gamma^\mu \Gamma(\epsilon) \left(\frac{1}{q^2}\right)^\epsilon$$

$$= i g_s \gamma^\mu T^a \frac{g_s^2}{16\pi^2} \frac{3}{2} C_2(G) \left(\frac{4\pi\mu^2}{q^2}\right)^\epsilon \Gamma(\epsilon)$$

 +  +  finite

For $\overline{MS}$ scheme.

$$\delta Z_F = -\frac{g_s^2}{16\pi^2} [C_2(r) + C_2(G)] \left[\frac{1}{\epsilon} - \gamma + \ln 4\pi\right]$$

summary;

$$\delta Z_A = \frac{g_s^2}{16\pi^2}\left[\frac{5}{3}C_2(G) - \frac{4}{3}n_f\, C(r)\right]\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi\right)$$

$$\delta Z_\psi = -\frac{g_s^2}{16\pi^2}C_2(r)\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi\right)$$

$$\delta Z_1 = -\frac{g_s^2}{16\pi^2}\left[C_2(r)+C_2(G)\right]\left[\frac{1}{\epsilon}-\gamma+\ln 4\pi\right]$$

$$Z_g = \frac{Z_1}{Z_\psi Z_A^{1/2}}$$

$$= \left[1-\frac{g_s^2}{16\pi^2}\left(C_2(r)+C_2(G)\right)\Delta'\right]\left[1+\frac{g_s^2}{16\pi^2}C_2(r)\Delta'\right]\left[1-\frac{1}{2}\frac{g_s^2}{16\pi^2}\left(\frac{5}{3}C_2(G)-\frac{4}{3}n_f\, C(r)\right)\Delta'\right]$$

$$= 1-\frac{g_s^2}{16\pi^2}\left[C_2(r)+C_2(G)-C_2(r)+\frac{1}{2}\left(\frac{5}{3}C_2(G)-\frac{4}{3}n_f\, C(r)\right)\Delta'\right]$$

$$= 1-\frac{g_s^2}{16\pi^2}\left[\frac{11}{6}C_2(G)-\frac{2}{3}n_f\, C(r)\right]\Delta'$$

with $\Delta' = \frac{1}{\epsilon}-\gamma+\ln 4\pi$

$\beta$ function:

note: $g_0 = \mu^{\epsilon} g\, Z_g$

$$\alpha_{s0} = (\mu^2)^{\epsilon} \alpha_s Z_g^2$$

since the $\alpha_{s0}$ is not depending on the scale:

$$\mu^2 \frac{d\alpha_{s0}}{d\mu^2} = 0$$

$$\Rightarrow \mu^2 \frac{d}{d\mu^2}\left[(\mu^2)^{\epsilon} \alpha_s Z_g^2\right]$$

$$= \mu^2\left[\epsilon(\mu^2)^{\epsilon-1} \alpha_s Z_g^2 + \mu^{2\epsilon} \frac{d\alpha_s}{d\mu^2} Z_g^2 + (\mu^2)^{\epsilon} \alpha_s \frac{dZ_g^2}{d\mu^2}\right]$$

$$= (\mu^2)^{\epsilon}\left[\epsilon \alpha_s Z_g^2 + \mu^2 \frac{d\alpha_s}{d\mu^2} Z_g^2 + \alpha_s \mu^2 \frac{d\alpha_s}{d\mu^2} \frac{dZ_g^2}{d\alpha_s}\right]$$

$$= (\mu^2)^{\epsilon}\left[\epsilon \alpha_s Z_g^2 + \beta(\alpha_s, \epsilon) Z_g^2 + 2\alpha_s \beta(\alpha_s, \epsilon) Z_g \frac{dZ_g}{d\alpha_s}\right]$$

$$= (\mu^2)^{\epsilon} Z_g\left[\beta(\alpha_s, \epsilon) Z_g + \epsilon \alpha_s Z_g + 2\alpha_s \beta(\alpha_s, \epsilon) \frac{dZ_g}{d\alpha_s}\right] = 0$$

$$\Rightarrow \left[\beta(\alpha_s, \epsilon) + \epsilon \alpha_s + 2\alpha_s \beta(\alpha_s, \epsilon) \frac{d}{d\alpha_s}\right] Z_g = 0$$

where: $\beta(\alpha_s, \epsilon) = \mu^2 \frac{d\alpha_s}{d\mu^2}$

$$Z_g = 1 + \sum_i \frac{Z^{(i)}}{\epsilon^i}$$

$$\beta(\alpha_s, \epsilon) = \beta(\alpha_s) + \sum_{i=1} \beta^{(i)}(\alpha_s) \epsilon^i$$

$$\Rightarrow \left\{\beta(\alpha_s) + \sum_{i\geq 1} \beta^{(i)}(\alpha_s) \epsilon^i + \epsilon \alpha_s + 2\alpha_s \left[\beta(\alpha_s) + \sum \beta^i(\alpha_s) \epsilon^i\right] \frac{d}{d\alpha_s}\right\}$$

$$\times \left\{1 + \sum_i \frac{Z^{(i)}}{\epsilon^i}\right\} = 0$$

$$\Rightarrow \begin{cases} \beta^i(\alpha_s) = 0 & i > 1 \\ \beta^1(\alpha_s) = -\alpha_s \\ \beta(\alpha_s) = 2\alpha_s^2 \frac{d}{d\alpha_s} Z^{(1)} \end{cases}$$

$$Z_g = 1 - \frac{g_s^2}{16\pi^2}\left[\frac{11}{6} C_2(G) - \frac{2}{3} n_f C(r)\right]\left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right)$$

$$Z^{(1)} = -\frac{\alpha_s}{4\pi}\left[\frac{11}{6} C_2(G) - \frac{2}{3} n_f C(r)\right]$$

$$\beta(\alpha_s) = 2\alpha_s^2 \frac{d}{d\alpha_s} Z^{(1)}$$

$$= -\frac{\alpha_s^2}{4\pi}\left[\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r)\right]$$

Thus, the one-loop beta function

$$\beta(\alpha_s) = b_0 \alpha_s^2 \quad \text{with} \quad b_0 = -\frac{1}{4\pi}\left[\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r)\right]$$

For $SU(N)$, ~~C(N)~~ the fundamental representation

$$C(N) = \frac{1}{2}, \quad C_2(N) = \frac{N^2-1}{2N}, \quad C_2(G) = C(G) = N$$

Thus: $\displaystyle\int_{\mu^2}^{Q^2} d\ln\mu^2 = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{d\alpha_s}{\beta}$

$$\Rightarrow \ln\frac{Q^2}{\mu^2} = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{d\alpha_s}{\beta} = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{d\alpha_s}{b_0 \alpha_s^2}$$

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 - b_0 \alpha_s(\mu^2) \ln\frac{Q^2}{\mu^2}}$$

$$= \frac{\alpha_s(\mu^2)}{1 + \frac{1}{4\pi}\left(11 - \frac{2}{3} n_f\right) \alpha_s(\mu^2) \ln\frac{Q^2}{\mu^2}}$$