03 Jun 2016 C. Lee

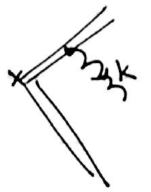
VII. Soft function : PT & NP

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = H(Q^2,\mu)\int dt_n\, dt_{\bar n}\, dk_s\; J_n(t_n,\mu)\, J_{\bar n}(t_{\bar n},\mu)\, S(k_s,\mu)$$

$$S(k_s,\mu) = \int dk_L\, dk_R\; S(k_L, k_R,\mu)$$

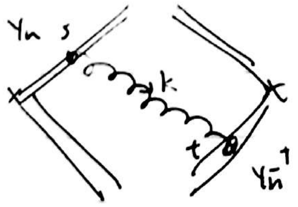
$$S(k_L,k_R,\mu) = \frac{1}{N_C}\mathrm{Tr}\sum_{X_s}\langle 0|\,\bar Y_{\bar n}^{\dagger}\, Y_n^{\dagger}\,|X_s\rangle\langle X_s|\,Y_n \bar Y_{\bar n}\,|0\rangle \times \delta\left(k_L - k^{+}_{(L)}(X_s)\right)\delta\left(k_R - k^{+}_{(R)}(X_s)\right)$$

Feyn rule :



$$\langle k|\, ig\int_0^\infty ds\; n\cdot A(ns)\,|0\rangle$$

$$ig$$



$$ig\int_0^\infty ds\; n\cdot A(ns)\;(-ig)\int_0^\infty dt\;\bar n\cdot A(\bar n t)$$

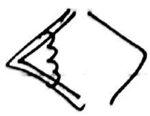
$$g^2\mu^{2\epsilon}\int d^Dk\;\frac{-i\,n\cdot\bar n}{k^2+i\epsilon}\int_0^\infty ds\int_0^\infty dt\; e^{-ik\cdot(ns-\bar n t)}$$

$$= -2ig^2\mu^{2\epsilon}\int\frac{d^Dk}{k^2+i\epsilon}\int_0^\infty ds\; e^{(-ik\cdot n-\epsilon)s}\int_0^\infty dt\; e^{(ik\cdot\bar n-\epsilon)t}$$

$$= -2ig^2\mu^{2\epsilon}\int\frac{d^Dk}{k^2+i\epsilon}\;\frac{-1}{-ik\cdot n-\epsilon}\;\frac{-1}{i\bar n\cdot k-\epsilon}$$

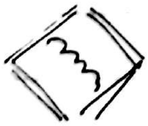
$$\times C_F \quad = 2ig^2\mu^{2\epsilon}\int d^Dk\;\underbrace{\frac{1}{k^2+i\epsilon}}_{\to\, -2\pi i\,\theta(k^0)\,\delta(k^2)}\;\frac{1}{n\cdot k - i\epsilon}\;\frac{1}{\bar n\cdot k + i\epsilon}$$

~~that~~ virtual:



$= 0$ in DR

Real:



$$+\text{h.c.} = 4g^2\mu^{2\epsilon}C_F\int d^Dk\;\frac{1}{k^+k^-}\;\theta(k^0)\,\delta(k^2)\,2\pi \times\left[\theta(k^- - k^+)\,\delta(k_s - k^+) + \theta(k^+ - k^-)\,\delta(k_s - k^-)\right]$$

$$= 4g^2\mu^{2\epsilon}C_F\int\frac{dk^+dk^-}{2(2\pi)^4}\int\frac{d^dk_\perp}{(2\pi)^d}\;\frac{1}{k^+k^-}\;\delta(k^+k^- - k_\perp^2)\left[\theta(k^- - k^+)\,\delta(k_s - k^+) + \theta(k^+ - k^-)\,\delta(k_s - k^-)\right] \times\theta(k^+ + k^-)$$

$k^+k^- > 0$

$k^+k^- > 0$

$$= \frac{2g^2\mu^{2\epsilon}}{(2\pi)^{D-1}} C_F \int_0^\infty dk^+ \int_0^\infty \frac{dk^-}{k^+k^-} \int \frac{dk_\perp^2}{2} \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} (k_\perp^2)^{-\epsilon}\, \delta(k^+k^- - k_\perp^2) \times \left[\theta(k^- - k^+)\,\delta(k_S - k^+) + \theta(k^+ - k^-)\,\delta(k_S - k^-) \right]$$

$$= \frac{g^2\mu^{2\epsilon}}{(2\pi)^{3-2\epsilon}} \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} C_F \int_0^\infty \frac{dk^+\,dk^-}{k^+k^-} (k^+k^-)^{-\epsilon}\, 2\cdot\theta(k^- - k^+)\,\delta(k_S - k^+)$$

$$= \frac{2g^2\mu^{2\epsilon}\pi^\epsilon 4^\epsilon}{4\pi^2} \frac{1}{\Gamma(1-\epsilon)} C_F \int_{k_S}^\infty dk^- \,(k^- k_S)^{-1-\epsilon}$$

$$= 2\frac{\alpha_s C_F}{\pi} \frac{(4\pi\mu^2)^\epsilon}{\Gamma(1-\epsilon)}\, k_S^{-1-\epsilon}\, \frac{1}{-\epsilon}\, k^{-\epsilon}\Big|_{k_S}^\infty$$

$$= 2\frac{\alpha_s C_F}{\pi} \frac{(4\pi\mu^2)^\epsilon}{\Gamma(1-\epsilon)} \frac{1}{\epsilon}\, \underbrace{k_S^{-1-2\epsilon}}_{-\frac{1}{2\epsilon}\delta(k_S) + \left[\frac{\theta(k_S)}{k_S}\right]_+ - 2\epsilon\left[\theta(k_S)\ln\right.}$$

$$\overset{\overline{MS}}{=} 2\frac{\alpha_s C_F}{\pi} \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \frac{1}{\epsilon}\, \underbrace{\left(\frac{k_S}{\mu}\right)^{-1-2\epsilon}}_{= -\frac{1}{2\epsilon}\delta\left(\frac{k_S}{\mu}\right) + \left[\frac{\theta(k_S/\mu)}{k_S/\mu}\right]_+ - 2\epsilon\left[\frac{\theta(k_S)\ln k_S/\mu}{k_S/\mu}\right]_+} \frac{1}{\mu}$$

$$= \frac{\alpha_s C_F}{\pi} \left\{ -\frac{1}{\epsilon^2}\delta(k_S) + \frac{2}{\epsilon}\frac{1}{\mu}\left[\frac{\theta(k_S)}{k_S/\mu}\right]_+ - \frac{4}{\mu}\left[\frac{\theta(k_S)\ln k_S/\mu}{k_S/\mu}\right]_+ + \frac{\pi^2}{12}\delta(k_S) \right\}$$

$\downarrow$

integrated $\quad S_c(k_S,\mu) = 1 + \frac{\alpha_s C_F}{\pi}\left\{ -\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{k_S}{\mu} - 2\ln^2\frac{k_S}{\mu} + \frac{\pi^2}{12} \right\}$

Z_S $\qquad\qquad$ S_{ren}

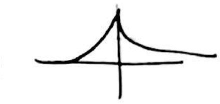
$\downarrow$

known to $\mathcal{O}(\alpha_s^2)$

$S(k_s,\mu) = \int dk' \; S_{PT}(k_s - k_s', \mu) \; f_{NP}(k_s')$

Q —— H
$Q\lambda$ —— J
$Q\lambda^2$ —— S
Λ_{QCD} - - - - NP

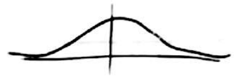
$S = \frac{1}{N_c} \mathrm{Tr} \sum_{X_s} \langle 0| Y_{\bar n}^\dagger Y_n |X_s\rangle\langle X_s| Y_n Y_{\bar n}^\dagger |0\rangle \; \delta(e - e(X_s))$



$\tau(X) = \sum_{i\in X} |\vec p_i^{\,T}| \, e^{-|\eta_i|}$

$\xrightarrow{\text{1D}} \hat e \int p_T \qquad \eta_i = \ln\cot\frac{\theta}{2}$

$e(X) = \sum_{i\in X} |\vec p_i^{\,T}| \, f(\eta_i)$



angularities $f = e^{-|\eta|(1-a)}$
broadening $f = 1$ (SCET$_{II}$)
"C-param" $f = \frac{3}{\cosh\eta}$ etc.

define an op. $\hat e|X\rangle = e(X)|X\rangle \qquad \mathcal{E}_T(\eta)|X\rangle = \sum_{i\in X}|\vec p_i^{\,T}|\,\delta(\eta-\eta_i)\,|X\rangle$

$\hat e = \int_{-\infty}^{\infty} d\eta \; f(\eta)\; \mathcal{E}_T(\eta)$

$\mathcal{E}_T(\eta) = \cancel{\lim_{R\to\infty} \int_0^\infty dt}$
$\frac{1}{\cosh^3\eta} \int_0^{2\pi} d\phi \; \lim_{R\to\infty} R^2 \int_0^\infty dt \; \hat n_i \, T_{0i}(t, R\hat n)$

$S = \frac{1}{N_c}\mathrm{Tr} \cancel{\sum_{X_s}} \langle 0| Y_{\bar n}^\dagger Y_n \; \delta(e-\hat e) \cancel{|X_s\rangle}\langle X_s| Y_n Y_{\bar n}^\dagger |0\rangle$

$\delta(e-\hat e) = \delta(e) + \hat e\,\delta'(e) + \dots$

$= \int_0^\Delta de \; \sigma(\dots) \to \Sigma(e) \qquad e\,\sigma'(e)$

$\to \frac{\Delta}{Q} \qquad \to \frac{1}{\Delta}$

1st at $0 \ll \Delta \ll 1$

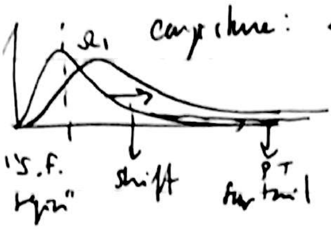
btw can use in QCD c.s. $\sum_X \langle 0|j^\dagger|X\rangle\langle X|j|0\rangle \to \langle 0|j^\dagger \delta(e-\hat e)\, j|0\rangle$

$\delta(e-\hat e)$

$T_{\mu\nu} = T^{n}_{\mu\nu} + T^{\bar n}_{\mu\nu} + T^{s}_{\mu\nu}$

to avoid $|X\rangle \to |X_n\rangle|X_{\bar n}\rangle|X_s\rangle$

$$\sigma = \int de' \, \frac{\sigma_{PT}(e-e')}{\sigma_{tot}} f_{NP}(e')$$



compare: $\langle f\rangle_{NP} = \frac{c_e \Omega_1}{a}$ e — universal for diff e

$c_e = \int d\eta \, f_e(\eta)$

$$\langle e\rangle = \int de \int de' \, e \, \sigma_n(e-e') f_{NP}(e') = \int de \int de' \, (e-e'+e') \, \sigma_{PT}(e-e') f_{NP}(e')$$

$$= \int de \int de' \left[e' \sigma_{PT}(e-e') f_{NP}(e') + (e-e') \sigma_{PT}(e-e') f_{NP}(e') \right]$$

$$\int de \, \sigma_{PT}(e) \int de' e' f_{NP}(e') \qquad \frac{1}{\sigma_{tot}} \int de \, e \, \sigma_{PT}(e) \int f_{NP}(e')$$

$$\sigma_{tot} \langle f\rangle \qquad\qquad \langle e\rangle_{PT} \cdot 1$$

$$= \langle e\rangle_{PT} + \langle f\rangle_{NP}$$

$$\langle e\rangle = \frac{1}{N_C} \mathrm{Tr} \, \langle 0| Y_{\bar n}^\dagger Y_n \, \hat e \, Y_n Y_{\bar n} |0\rangle$$

$$= \frac{1}{N_C} \mathrm{Tr} \, \cancel{\frac{1}{\cosh^2\eta} \int_0^{2\pi} \frac{d\phi}{\ldots} \ldots} \, \langle 0| Y_n^\dagger Y_{\bar n}^\dagger \frac{1}{a} \int d\eta \, f(\eta) \, \hat{\mathcal{E}}_T(\eta) \, Y_n Y_{\bar n} |0\rangle$$

$\Lambda_{\eta'}^{-1}\Lambda_{\eta'}$ Lorentz boost by η'

$$\Lambda_{\eta'}|0\rangle = |0\rangle$$

$$\Lambda_{\eta'}^{-1} Y_n \Lambda_{\eta'} = P\exp\left[ig \int_{-\infty}^0 ds \, e^{\eta'} \hat n \cdot A(\hat n s \, e^{\eta'}) \right] \qquad s' = e^{\eta'} s$$

$$= Y_n$$

$$\Lambda_{\eta'} \hat{\mathcal{E}}_T(\eta) \Lambda_{\eta'} = \hat{\mathcal{E}}_T(\eta+\eta')$$

$$= \frac{1}{N_C} \mathrm{Tr} \, \frac{1}{a} \underbrace{\int d\eta \, f_e(\eta)}_{c_e} \, \langle 0| Y^\dagger Y^\dagger \, \hat{\mathcal{E}}(\eta+\eta') \, YY |0\rangle$$

choose $\eta+\eta' = 0$ (or 17, ...)

$$= \frac{c_e \Omega_1}{a} \quad \text{where } \Omega_1 = \frac{1}{N_C} \mathrm{Tr} \langle 0| Y^\dagger Y^\dagger \, \hat{\mathcal{E}}_T(0) \, YY |0\rangle$$

1^{st} def. of Ω_1 to all orders.

also DIS event shapes.