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VIII RGE & resummation 03 Jun 2016

$$\frac{d\sigma}{d\tau} = \sigma_0 H(Q^2,\mu) \int dt_n\, dt_{\bar n}\, dk_s\, J_n(t_n,\mu) J_{\bar n}(t_{\bar n},\mu) S(k_s,\mu) \times \delta\left(\tau - \frac{t_n + t_{\bar n}}{Q^2} - \frac{k_s}{Q}\right)$$

$$|C|^2 = H(Q^2,\mu) = 1 - \frac{\alpha_s C_F}{2\pi}\left(8 - \frac{7\pi^2}{6} + \ln^2\frac{\mu^2}{Q^2} - 3\ln\frac{Q^2}{\mu^2}\right)$$

Current

$$\cancel{S_c(k,\mu) = 1 + \frac{\alpha_s C_F}{4\pi}\left(-8\ln^2\frac{k}{\mu} + \frac{\pi^2}{3}\right)}$$

$$S(k,\mu) = \delta(k)\left[1 + \frac{\alpha_s C_F}{4\pi}\frac{\pi^2}{3}\right) - \frac{\alpha_s C_F}{4\pi}\frac{16}{\mu}\left[\frac{\theta(k_s/\mu)\ln k_s/\mu}{k_s/\mu}\right]_+ \qquad \mathcal{L}_1(k_s/\mu)$$

$$J(t,\mu) = \delta(t)\left(1 + \frac{\alpha_s C_F}{4\pi}(7-\pi^2)\right) + \frac{\alpha_s C_F}{4\pi}\left[-\frac{3}{\mu^2}\mathcal{L}_0(t/\mu^2) + \frac{4}{\mu^2}\mathcal{L}_1(t/\mu)\right]$$

easier to work with Laplace transforms
(or Fourier...)

$$\tilde\sigma(\nu) = \int_0^\infty d\tau\, e^{-\nu\tau}\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} \equiv \mathcal{L}\{\sigma\}$$

$$= H(Q^2,\mu)\,\tilde J_n(Q^2/\nu,\mu)\,\tilde J_{\bar n}(Q^2/\nu,\mu)\,\tilde S(Q/\nu,\mu)$$

$$\mathcal{L}\{1\} = 1$$
$$\mathcal{L}\{\mathcal{L}_0(\tau)\} = \ln \nu e^{\gamma_E}$$
$$\mathcal{L}\{\mathcal{L}_1(\tau)\} = \frac{1}{2}\left(\ln^2 \nu e^{\gamma_E} + \frac{\pi^2}{6}\right)$$
etc. $\quad \ln \nu$

$$\Rightarrow \quad \tilde S(Q/\nu,\mu) = 1 + \frac{\alpha_s C_F}{4\pi}\frac{\pi^2}{3} - \frac{\alpha_s C_F}{4\pi}\left(8\ln^2\left(\frac{Q}{\mu\nu}\right) + 8\frac{\pi^2}{6}\right)$$
$$= 1 + \frac{\alpha_s C_F}{4\pi}\left(-8\ln^2\frac{Q}{\mu\nu} - \pi^2\right)$$

$$\tilde J(Q^2/\nu,\mu) = 1 + \frac{\alpha_s C_F}{4\pi}\left(7 - \pi^2 - 3\ln\frac{Q^2}{\mu^2\nu} + 2\left(\ln^2\frac{Q^2}{\mu^2\nu} + \frac{\pi^2}{6}\right)\right)$$
$$= 1 + \frac{\alpha_s C_F}{4\pi}\left(2\ln^2\frac{Q^2}{\mu^2\nu} - 3\ln\frac{Q^2}{\mu^2\nu} + 7 - \frac{2\pi^2}{3}\right)$$

$$\cancel{\tilde\sigma(\nu) = 1 + \alpha_s(\ln^2\nu + \ln\nu + 1) + \alpha_s^2(\ln^4\nu + \ln^3\nu + \cdots)}$$

$H, \tilde{J}, \tilde{S}$ all take the form

$$F(Q^j, \mu) = 1 + \frac{\alpha_s C_F}{4\pi}\left[\frac{\Gamma_F^0}{j^2} \ln^2 \frac{Q^j}{\mu^j \bar{\nu}} - \frac{\gamma_J^0}{j} \ln \frac{Q^j}{\mu^j \nu} + e_F \right]$$

where $j = 2$ for J, $j = 1$ for H, S

$\nu = 1$ for H

$$\begin{aligned} \Gamma_H^0 &= -8 C_F & \gamma_H^0 &= -12 C_F \\ 2\times \quad \Gamma_J^0 &= 8 C_F \quad \times 2 & \gamma_J^0 &= 6 C_F \quad \times 2 \\ \Gamma_S^0 &= -8 C_F & \gamma_S^0 &= 0 \\ \hline & \quad 0 & & \quad 0 \end{aligned}$$

$$\mu \frac{d}{d\mu} \tilde{F} = \left(\frac{2\Gamma_F^0}{j} \ln \frac{\mu^j \bar{\nu}}{Q^j} + \gamma_F^0 \right) \tilde{F}$$

$\downarrow$ all orders

$$\gamma_{\tilde{F}}(\mu) \tilde{F}$$

$$= \frac{2}{j} \Gamma_F^{\text{cusp}}[\alpha_s] \ln \frac{\mu^j \nu e^{\gamma_E}}{Q^j} + \gamma_{\tilde{F}}[\alpha_s]$$

$$\Gamma_F^{\text{cusp}}[\alpha_s] = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \Gamma_F^n \qquad \gamma_{\tilde{F}}[\alpha_s] = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \gamma_F^n$$

$$\Gamma_F^{\text{cusp}} = -\frac{j_F \kappa_F}{2} \Gamma_{\text{cusp}} \rightarrow$$

$\theta \to \pi$

$\downarrow$ known to 3 loops

$$\tilde{F}(Q^j \nu, \mu) = F(Q^j \nu, \mu_0) \underbrace{\exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_{\tilde{F}}(\mu') \right]}_{U_F(\mu, \mu_0)}$$

$$U_F(\mu,\mu_0) = \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left(\gamma \; -\kappa_F \Gamma_{cusp}[\alpha_s] \ln \frac{\mu'^j \bar{\nu}}{Q^j} \; + \gamma_F[\alpha_s] \right) \right]$$

$$j\left(\ln \frac{\mu}{\mu_0} + \ln \frac{\mu_0 \bar{\nu}^{1/j}}{Q} \right)$$

$$= \exp\left(-j\kappa_F K_\Gamma(\mu,\mu_0) - \kappa_F \eta_\Gamma(\mu,\mu_0) \ln \frac{\mu_0^j \bar{\nu}}{Q^j} + K_{\gamma_F}(\mu,\mu_0) \right)$$

$$K_\Gamma = \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma_{cusp}[\alpha_s] \ln \frac{\mu'}{\mu_0}$$

$$\eta_\Gamma = \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Gamma_{cusp}[\alpha_s] \qquad K_\gamma = (\eta_\Gamma)(\Gamma \to \gamma)$$

$$U_F = e^{-j\kappa_F K_\Gamma + K_\gamma} \left(\frac{\mu_0^j \bar{\nu}}{Q^j} \right)^{-\kappa_F \eta_\Gamma}$$

$\downarrow$ note if pick $\mu_0 = Q_0 \bar{\nu}^{1/j}$

$e^{-j\kappa_F K_\Gamma + K_\gamma} \to$ "exponentiates"

- - - - μ
Q ——— μ_H
$Q/\bar{\nu}^{1/2}$ ——— μ_J
$Q/\bar{\nu}$ ——— μ_S

$$\tilde{\sigma}(\nu) = \tilde{\sigma}(\nu, \mu_{H,J,S})\, e^{-K_H - 2K_J - K_S} \qquad K_F = (-j\kappa_F K_\Gamma + K_\gamma)(\mu, \mu_F)$$

$$\sim \exp\Big\{ \alpha_s (\ln^2 \nu + \ln \nu + 1)$$
$$+ \alpha_s^2 (\ln^3 \nu + \ln^2 \nu + \ln \nu + 1)$$
$$+ \alpha_s^3 (\ln^4 \nu + \ln^3 \nu + \ln^2 \nu + \ln \nu + 1) + \cdots \Big\}$$

large: $\ln \nu \sim \frac{1}{\alpha}$ $\qquad$ LL $\sim 1/\alpha$ $\quad$ NLL ~ 1 $\quad$ N^{2}LL $\sim \alpha$ $\quad$ N^{3}LL $\hookrightarrow$ hardest in SCET

N^{k}LL counting naturally organized in exponent of $\tilde{\sigma}(\nu)$

$$\tilde{\sigma}(\nu) = \underbrace{H(\mu_H)\, \tilde{J}^2\Big(\underset{\downarrow \, \ln \frac{Q^2}{\nu}}{\tfrac{Q^2}{\nu}}, \mu_J\Big)\, \tilde{S}\Big(\underset{\downarrow \, \ln \frac{Q}{\nu}}{\tfrac{Q}{\nu}}, \mu_S\Big)}\, e^{-K_H - 2K_J - K_S} \left(\frac{\mu_H}{Q}\right)^{-\eta_H} \left(\frac{\mu_J^2 \bar{\nu}}{Q^2}\right)^{-2\eta_J} \left(\frac{\mu_S \nu}{Q}\right)^{-\eta_S}$$

$$= H(\mu_H)\, \tilde{J}^2(\partial_{\eta_J}, \mu_J)\, \tilde{S}(\partial_{\eta_S}, \mu_S) \qquad \nu^{-\eta} \to \frac{e^{-\gamma_E \eta}}{\Gamma(\eta)} \;\#\; \tau^{\eta}$$

$$\sigma_c(\tau) = H(\mu_H)\, \tilde{J}(\partial_{\eta_J}, \mu_J)\, \tilde{S}(\partial_{\eta_S}, \mu_S)\, e^{-K_H - 2K_J - K_S} \left(\frac{\mu_H}{Q}\right)^{-\eta_H} \left(\frac{\mu_J^2}{Q^2 \tau}\right)^{-2\eta_J} \left(\frac{\mu_S}{Q\tau}\right)^{\eta_S}$$

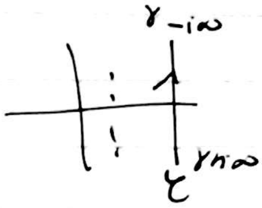
$$\sigma = \frac{d\sigma_c}{d\tau} \qquad\qquad \times \frac{e^{-\gamma_E \eta}}{\Gamma(1+\eta)}$$

↓

can choose scales in mom. space

otherwise run into spurious poles

in $\mathcal{L}^{-1}\{\tilde{\sigma}(\nu)\} = \int_\gamma \frac{d\nu}{2\pi i} e^{\nu\tau} \tilde{\sigma}(\nu)$



$$\begin{array}{ccc} \tilde{\sigma}(\nu, \mu_{H,J,S}) & \xrightarrow{\mathcal{L}^{-1}} & \sigma(\tau, \mu_{H,J,S}) \\ \mu_H = Q,\ \mu_{J,S} = \frac{Q}{\nu^{1/j}} \Big\downarrow \text{set scales} & & \Big\downarrow \text{set scales}\ \ \mu_H = Q,\ \mu_{J,S} = Q\tau^{1/j} \\ \tilde{\sigma}_{\text{resum}}(\nu) & \xrightarrow{\mathcal{L}^{-1}} & \sigma_{\text{resum}}(\tau) \end{array}$$

commute if to $O(\alpha_s^\infty)$, not at finite order.

explicit form:

$$\mu \frac{d\alpha_s}{d\mu} = \beta[\alpha_s] \qquad \Rightarrow \frac{d\mu}{\mu} = \frac{d\alpha}{\beta}$$

$$= -2\alpha_s \sum_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \beta_n$$

$$= -2\frac{\alpha_s^2}{4\pi}\left(\beta_0 + \frac{\alpha_s}{4\pi}\beta_1 + \cdots\right)$$

$$\alpha_s(\mu) = \alpha_s(\mu_0)\left\{X + \alpha_s(\mu_0)\frac{\beta_1}{4\pi\beta_0}\ln X + \cdots\right\}^{-1}$$

← go to K_Γ first

$$X = 1 + \frac{\alpha_s(\mu_0)}{2\pi}\beta_0 \ln\frac{\mu}{Q}$$

$$\Rightarrow \eta_\Gamma = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]} \Gamma_{\text{cusp}}[\alpha]$$

$$\eta_\Gamma^{LL} = \int_{\alpha_0}^{\alpha_s} \frac{d\alpha}{-\frac{\alpha^2}{2\pi}\beta_0 + \cdots}\left(\frac{\alpha}{4\pi}\Gamma_0 + \cdots\right) = \frac{-\Gamma_0}{2\beta_0}\int_{\alpha_0}^{\alpha_s}\frac{d\alpha}{\alpha} = \frac{-\Gamma_0}{2\beta_0}\ln\underbrace{\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}}_{r} = \frac{-\Gamma_0}{2\beta_0}\ln r$$

$$\eta_\Gamma^{NLL} = \int_{\alpha_0}^{\alpha_s}\frac{d\alpha}{-\frac{\alpha^2}{2\pi}\beta_0 - \frac{\alpha^3}{8\pi^2}\beta_1}\left(\frac{\alpha}{4\pi}\Gamma_0 + \frac{\alpha^2}{16\pi^2}\Gamma_1\right) = -\frac{\Gamma_0}{2\beta_0}\int_{\alpha_0}^{\alpha_s}\frac{d\alpha}{\alpha}\left(\frac{1 + \frac{\alpha}{4\pi}\frac{\Gamma_1}{\Gamma_0}}{1 + \frac{\alpha}{4\pi}\frac{\beta_1}{\beta_0}}\right)$$

$$= -\frac{\Gamma_0}{2\beta_0}\int_{\alpha_0}^{\alpha_s}\frac{d\alpha}{\alpha}\left(1 + \frac{\alpha}{4\pi}\left(\frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0}\right)\right)$$

$$= \eta_{LL} - \frac{\Gamma_0}{2\beta_0}\left(\frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0}\right)\frac{1}{4\pi}\left(\alpha_s(\mu) - \alpha_s(\mu_0)\right)$$

$$\eta_{NLL} = -\frac{\Gamma_0}{2\beta_0}\frac{\alpha_s(\mu_0)}{4\pi}\left(\frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0}\right)(r-1)$$

etc.

$$K_\Gamma^{\mu} = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]}\, \Gamma_{cusp}[\alpha] \int_{\alpha_s(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta[\alpha']}$$

$$= \int_{\alpha_0}^{\alpha_s} \frac{d\alpha}{-\frac{\alpha^2}{2\pi}\beta_0}\, \frac{\alpha}{4\pi}\Gamma_0 \int_{\alpha_0}^{\alpha} \frac{d\alpha'}{-\frac{\alpha'^2}{2\pi}\beta_0}$$

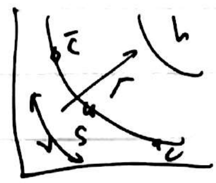
$$= \frac{\pi\Gamma_0}{\beta_0^2} \int_{\alpha_0}^{\alpha_s} \frac{d\alpha}{\alpha}\left(\frac{1}{\alpha_0} - \frac{1}{\alpha}\right)$$

$$= \frac{\pi\Gamma_0}{\beta_0^2}\, \frac{1}{\alpha_s(\mu_0)} \left[\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} - \alpha_s(\mu_0)\left(\frac{1}{\alpha_s(\mu_0)} - \frac{1}{\alpha_s(\mu)}\right)\right]$$

$$= \frac{\pi\Gamma_0}{\beta_0^2}\, \frac{1}{\alpha_s(\mu_0)} \left[\ln r - 1 + \frac{1}{r}\right] \qquad \sim \frac{1}{\alpha}$$

$\Rightarrow$ cross-section expressed as $\exp\left\{ \text{ratios of } \frac{\alpha_s(\mu)}{\alpha_s(\mu_{H,J,S})} \right\}$

$SCET_{II}$:



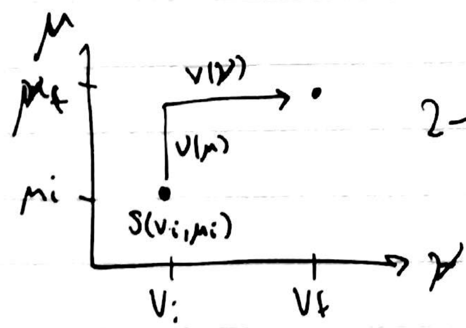
$J(\mu)\, S(\mu)$
divergent (rapidity logs)

need ν regulator:

$$W_n \to P\exp\left[ig \left(\frac{|\bar{n}\cdot P|}{\nu}\right)^{-\eta} \int_{-\infty}^{0} ds\, \bar{n}\cdot A_n\right]$$

$$S_n \to P\exp\left[ig \left(\frac{|2P_z|}{\nu}\right)^{-\eta/2} \int ds\, n\cdot A_s\right]$$

breaks boost inv., regulates rap. div.



2-D RG!

$SCET_+$ — jet substructure, jet algs.

$SCET_G$ — heavy ion, small-x fact. and.