Estimating Relative Luminosity for RHIC Spin Physics

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Abstract

Relative luminosities for different beam spin states must be determined to better than 10^{-4} – 10^{-3} in order to measure spin asymmetries to the desired accuracy at RHIC. It is demonstrated that biases due to high rates in luminosity monitors can be kept acceptably small for anticipated RHIC operating conditions. Additionally, the distribution of the estimates is shown to be approximately Gaussian (Normal) with known mean and standard error, permitting the construction of confidence intervals for the true spin asymmetry.

1 Introduction

Spin physics asymmetries are being measured at the RHIC polarized proton collider at the Brookhaven National Laboratory. These spin asymmetries, such as A_N,

$$A_N \;=\; \frac{1}{P_B} \cdot \frac{N_\uparrow/L_\uparrow \;-\; N_\downarrow/L_\downarrow}{N_\uparrow/L_\uparrow \;+\; N_\downarrow/L_\downarrow} \tag{1}$$

are frequently expected to be small, of order 0.01. In Eq. (1), P_B is the polarization of one of the beams; $N_\uparrow$, $N_\downarrow$ are the numbers of events for a particular physics process detected to the "left" of the beam; and $L_\uparrow$, $L_\downarrow$ are integrated beam luminosities for beam spin directions up ($\uparrow$) and down ($\downarrow$). Only relative luminosities are important in Eq. (1). It is often desired to measure such spin asymmetries to a relative uncertainty of a few percent. As a consequence, $L_\uparrow$

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and $L_\downarrow$ must be determined to order 10^{-4} – 10^{-3} so as to make a negligible contribution to the uncertainty on A_N, though averaging over many runs may allow a less stringent condition.

During operation in 2002 and 2003, the beam usually occurred in short (∼nsec) bunches spaced at 213 nsec, consisting of 55 filled and 5 empty bunches around each of the RHIC rings. For the STAR detector [1], two arrays of beam-beam counters on opposite ends of STAR were used to monitor the luminosity. Each array consisted of a set of scintillation counters covering pseudo-rapidity ($\mid \eta \mid = \mid \ln\,[\tan\,(\theta_{lab}/2)]\mid$) between approximately 3.3 and 5.0. They are similar in design to those used initially at the CDF and D0 detectors [2,3].

A luminosity event was recorded in a scaler whenever there was a coincidence between at least one beam-beam counter from the array on each end of STAR. Given the beam-beam counter acceptance, the cross section for the luminosity events at $\sqrt{s} = 200$ GeV is roughly 26 mb [4], or about half the pp total cross section. At a luminosity of 2×10^{31} /cm^2/sec achieved during the 2003 RHIC run, the average fraction of the colliding bunches that gave a luminosity event was $\lambda \sim 0.1$. In this scheme, multiple luminosity events from the passage of a single bunch (in each beam) through STAR could not be distinguished from single events because of the short bunch length, which leads to an error in the derived relative luminosity significantly larger than 10^{-4} – 10^{-3}.

This note describes a method to correct the number of luminosity events and places limits on possible biases. However, there are many additional issues that will require study and possible corrections in order to achieve reliable luminosity monitoring. It will be assumed that background events, for example from beam-gas interactions, can be accurately subtracted to better than 10^{-4} – 10^{-3}. The cross section measured by the beam-beam counters will be assumed to be spin independent. Since the method relies on an accurate determination of the fraction of bunches with no luminosity event, corrections will be needed for accidental coincidences. It will also be assumed that bunch to bunch phase space differences do not significantly affect the luminosity monitors. Finally, if there are sizable luminosity changes during a run, the method may need to be applied to several time intervals within the run to achieve a sufficiently accurate luminosity determination.

2 Method

The number of true luminosity events, n, for each bunch passage through the detector follows a Poisson distribution with probability

$$Pr(n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

and mean or expectation E[n] $= \langle n \rangle = \lambda$. The analysis that follows will concentrate on counts for a particular bunch; corrections should be applied before summing over bunches. In a particular run, each bunch will pass through the detector N times, and the desired integrated luminosity is proportional to $N\lambda$. Since a particular bunch passes through STAR every (213 nsec)(60 bunches) = 12.8 μsec, the value of N is of order 10^7 - 10^8 per run.

As previously noted, only a zero or one is recorded for each bunch passage in this scheme, where a one indicates that one or more events occurred. These outcomes have probabilities

$$\begin{aligned} Pr(1) &= \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} = 1 - e^{-\lambda} \\ Pr(0) &= \frac{\lambda^0}{0!} e^{-\lambda} = e^{-\lambda}. \end{aligned}$$

This implies that each bunch passage follows an independent Bernoulli distribution. Thus, the summed events, S, for a particular bunch in a run will have a binomial distribution

$$Pr(S = s) = \frac{N!}{s!(N-s)!} \left(1 - e^{-\lambda}\right)^s \left(e^{-\lambda}\right)^{N-s}$$

with mean and standard deviation

$$\mu_S = N\left(1 - e^{-\lambda}\right) \tag{2}$$

$$\sigma_S = \sqrt{Ne^{-\lambda}(1 - e^{-\lambda})}. \tag{3}$$

Using S as the relative integrated luminosity leads to a systematic error, since from Eq. (2)

$$\mu_S = N\lambda\left(1 - \lambda/2! + \lambda^2/3! - \lambda^3/4! + ...\right),$$

whereas $N\lambda$ is the desired quantity. Therefore, the relative systematic error (or relative bias) is approximately $\lambda/2!$, or much larger than desired (0.05 vs. 10^{-4} - 10^{-3}, when $\lambda = .1$).

It has been suggested [3] to use the expression

$$L \propto -N \ln(1 - S/N) \equiv \Lambda \tag{4}$$

for the relative integrated luminosity, since from Eq. (2)

$$-N \ln(1 - \mu_S/N) = N\lambda.$$

This estimator has the additional property that it is the maximum likelihood estimator (MLE) of $N\lambda$, the parameter value at which the observed count is most likely. This can be shown since

$$\ln\{Pr(S)\} = \ln\left(\frac{N!}{S!(N-S)!}\right) + S\ln\left(1 - e^{-\lambda}\right) - (N-S)\lambda,$$

and

$$\frac{\partial}{\partial\lambda}\ln\{Pr(S)\} = \frac{Se^{-\lambda}}{1 - e^{-\lambda}} - (N-S).$$

Setting this derivative equal to zero and solving gives

$$\lambda_{\text{MLE}} = -\ln\left(1 - \frac{S}{N}\right).$$

However, Λ is <u>not</u> an unbiased estimator for $N\lambda$ because

$$\text{E}[\Lambda] = \infty.$$

The mean is infinite since there is a very small, but nonzero, probability for a count to be recorded for every bunch passage, or $S = N$. Since Λ is not an unbiased estimator, it is important to investigate possible deviations from the desired quantity $N\lambda$. It might be expected that for sufficiently small λ, such deviations are negligible, and Eq. (4) can be used to obtain the relative luminosity. This is true, and will be demonstrated below.

Under mild regularity conditions [5], which are satisfied in this case, the MLE is asymptotically unbiased. Furthermore, the distribution of the quantity $\sqrt{N}(\lambda_{MLE} - \lambda)$ approaches a Gaussian distribution with mean zero and variance $1/I(\lambda)$ in the limit as N increases to infinity:

$$\sqrt{N}(\lambda_{\text{MLE}} - \lambda) \xrightarrow{d} \text{Normal}\left(0, \frac{1}{I(\lambda)}\right),$$

where $I(\lambda)$ is the Fisher Information,

Table 1
Results of simulations to estimate the relative bias as a function of the average fraction of colliding beam bunches that gave a luminosity event, λ. The observed mean and standard deviations are shown, along with the theoretical values from the asymptotic distribution, μ_Λ and σ_Λ, respectively. The relative bias is the fractional difference between the observed and expected means; the theoretical standard errors of the relative biases are also provided. The bias is less than 10^{-4} for $\lambda \leq 10$, and is consistent with zero for $\lambda \leq 1$.

λ	Obs. Mean	μ_Λ	St. Dev.	σ_Λ	Rel. Bias
15	1501675678	1.5×10^9	18576636	18080421	$(1.117 \pm 0.012) \times 10^{-3}$
13	1300234352	1.3×10^9	6679085	6651409	$(1.80 \pm 0.05) \times 10^{-4}$
10	1000013483	10^9	1485115	1484098	$(1.35 \pm 0.15) \times 10^{-5}$
1	99999990.7	10^8	13096	13108	$-(0.93 \pm 1.31) \times 10^{-7}$
.1	9999997.4	10^7	3244.7	3243	$-(2.6 \pm 3.2) \times 10^{-7}$
.01	1000000.54	10^6	1003.5	1002.5	$(5.4 \pm 10.0) \times 10^{-7}$
.001	100000.30	10^5	316.4	316.3	$(3.0 \pm 3.2) \times 10^{-6}$
.0001	9999.961	10^4	100.1	100.0	$-(3.9 \pm 10.0) \times 10^{-6}$

$$\begin{aligned} I(\lambda) &= -\frac{1}{N}\mathrm{E}\left[\frac{\partial^2}{\partial\lambda^2}\ln\{Pr(S)\}\right] \\ &= \frac{1}{N}\mathrm{E}\left[\left(\frac{\partial}{\partial\lambda}\ln\{Pr(S)\}\right)^2\right] \\ &= \frac{1}{e^\lambda - 1}. \end{aligned}$$

Thus, for N sufficiently large, $N\lambda_{\mathrm{MLE}} = \Lambda$ is approximately Gaussian distributed with mean

$$\mu_\Lambda = N\lambda \tag{5}$$

and standard deviation

$$\sigma_\Lambda = \sqrt{N\,(e^\lambda - 1)}. \tag{6}$$

Note that the maximum value of $\mu_\Lambda/\sigma_\Lambda$ from Eqs. (5,6) occurs for $\lambda \cong 1.6$, when approximately 20 % of the bunch passages give zero luminosity events.

In order to demonstrate that N on the order of $10^7 - 10^8$ is sufficiently large to ensure that the asymptotic distribution is achieved, simulations for a variety of values of λ were performed. For each of eight values of λ in the range 10^{-4}

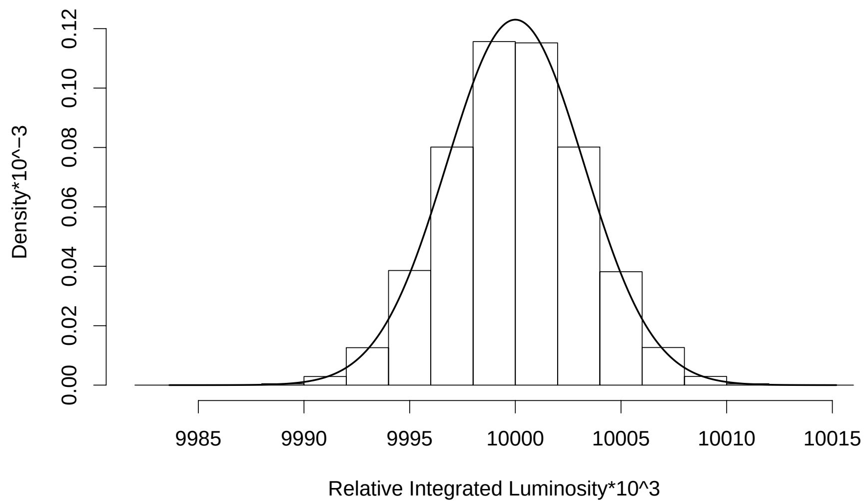

Fig. 1. This plot contains the histogram of Λ when $\lambda = .1$ for one million samples of size $N = 10^8$. Superimposed upon the plot is the theoretical Gaussian distribution for these N and λ. The two distributions show good agreement for this sample size.

to 15, one million samples of size $N = 10^8$ were simulated. For each sample, Λ was calculated, and the bias and variance of the estimator were determined. The results are included in Table 1, along with the theoretical values predicted by the asymptotic distribution.

Notice that as the value of λ increases above $\lambda = 10$, the magnitude of the bias increases rapidly; for values of $\lambda > 15$, samples with $S = N$ are often observed, causing severe biases. Additionally, to evaluate the asymptotic normality of the estimates, a histogram of the distribution of Λ is constructed for each value of λ in table 1, using 100,000 samples of size $N = 10^8$. The theoretical Gaussian distribution is superimposed and shows excellent agreement with the observed distribution for $\lambda \leq 1$. The plot with $\lambda = .1$ is displayed in Fig. 1.

The approximate normality of the parameters allows for the easy construction of a $100(1-\alpha)\%$ confidence interval for $N\lambda$. The form of this confidence interval is

$$(A, B) = (\Lambda - Z_{\alpha/2}\sigma_\Lambda, \Lambda + Z_{\alpha/2}\sigma_\Lambda), \tag{7}$$

where $Z_{\alpha/2}$ is the $100(1-\alpha/2)\%$ percentile of a standard normal distribution. Now, letting $(A_\uparrow, B_\uparrow)$ be the confidence interval for $N_\uparrow\lambda_\uparrow$ and $(A_\downarrow, B_\downarrow)$ be the confidence interval for $N_\downarrow\lambda_\downarrow$, and assuming that the two quantities are independent, a conservative $100[(1-\alpha)^2]\%$ confidence interval for the spin asymmetry can be constructed. This interval will have the form

$$(A_N(L_\uparrow = B_\uparrow, L_\downarrow = A_\downarrow), A_N(L_\uparrow = A_\uparrow, L_\downarrow = B_\downarrow)) ,$$

where $A_N(L_\uparrow, L_\downarrow)$ is as defined in Eq. (1). To construct a 95% confidence interval for the spin asymmetry, 97.47% confidence intervals for the individual $N\lambda$s are needed. This corresponds to $Z_{\alpha/2} = 2.236$ for construction of the confidence intervals in Eq. (7).

3 Discussion

The expression for the relative luminosity in Eq. (4) has been shown to have a fractional bias or error smaller than 10^{-4} for the mean number of detected interactions per bunch passage $\lambda \leq 10$. In this expression, the argument of the logarithm, $(1 - S/N)$, is the fraction of bunch passages with no luminosity event. In addition, Λ is the maximum likelihood estimator, which is asymptotically unbiased and has an approximate Gaussian distribution with mean $N\lambda$. Therefore, using Eq. (4) with the STAR beam-beam counter data for $N \sim 10^8$ will meet the requirements for monitoring luminosity for spin parameter measurements at RHIC, where luminosities up to $2 \times 10^{32}/\text{cm}^2/\text{sec}$ $(\lambda \sim 1)$ are projected. Additionally, confidence intervals can be constructed for both the luminosities and the spin asymmetry, allowing the uncertainty in these quantities to be assessed.

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References

[1] K.H. Ackermann et al., Nucl. Instr. and Meth. A 499 (2003) 624.

[2] D. Cronin-Hennessy, A. Beretvas, and P.F. Derwent, Nucl. Instr. and Meth. A 443 (2000) 37.

[3] J. Bantly et al., Fermilab technical report Fermilab-TM-1995 (June 1997, unpublished).

[4] J. Adams et al., submitted to Phys. Rev. Lett. and nucl-ex/0310058.

[5] Lehmann, E.L. and Casella, G.: *Theory of Point Estimation*, Springer Verlag (1988).